SOME PERTURBATION PROBLEMS IN THE THEORY OF DIFFERENTIAL AND INTEGRAL EQUATIONS

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This thesis is dedicated

to

my mother

and to

the revered teacher

Sri K.N. Seshagiriah

CERTIFICATE

This is to certify that the work embodied in the thesis entitled "Some Perturbation Problems in the theory of differential and integral equations" being submitted by Raghavendra V. has been carried out under my supervision and that this has not been submitted elsewhere for the award of any degree.

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CONTENTS

SYNOPSIS

In a very general sense, differential and integral equations arise in such diverse areas as physical, biological, oceanographic and engineering sciences. The manner in which such equations arise and their importance to various physical phenomena have been investigated by many scientists. However untill quite recently, attempts have not been made to develope and unify the theory of perturbations such as the theory of ordinary differential equations with impulsive perturbations and integral equations with discontinuous perturbations. The objective of this investigation is to study the existence, uniqueness and some stability properties of solutions of ordinary differential equations with impulsive perturbations and volterra integral equations with continuous and discontinuous perturbations.

The present thesis is divided into six chapters. The first chapter is devoted to introduction and to the outline of the thesis. Chapter 2 deals with preliminaries and basic results such as definition of various spaces with norms, admissibility of integral operators, generalized integral inequalities, distributions, distributional derivatives and integral representation of solutions of a measure differential equation through a given point.

In chapter 3, we consider the following system of integral equations

$$x(t) = f(t) + \int_0^t a(t,s)\, x(s)\, ds \tag{1}$$

and

$$x(t) = f(t) + \int_0^t a(t,s)\{x(s)+g(s,x(s))\}ds + \int_0^t b(t,s)h(s,x(s))ds \tag{2}$$

where x, f, g and h are elements of n-dimensional Euclidean space R^n, $a(t,s)$, $b(t,s)$ are $n \times n$ matrices defined for $0 \leq s \leq t < \infty$. Generally, the properties of solutions of nonlinear system (2) are frequently studied in comparison with the properties of solution of the linear system (1) particularly when the nonlinear functions $g(t,x)$ and $h(t,x)$ in (2) are either small as compared to x for sufficiently small x or small for sufficiently large t and for all $|x| < \infty$, where $|\,|$ denotes any vector norm in R^n. The principal interest that arises in this connection is concerned with the following question: whether the properties of solutions of (2) are shared with the corresponding properties of (1) when the perturbations satisfy certain more general conditions. In chapter 3, we shall compare in various ways the solution of (2) with the solution of the (unperturbed) linear system (1). Roughly speaking, we investigate sufficient conditions in order that a certain stability of the unperturbed equation (1) implies a corresponding local stability of equation (2). Our methods rest on the use of the resolvent kernel $R_1(t,s)$ for the linear system (1) and the familiar contraction principle. Our assumptions on the resolvent kernel may be viewed as sufficient conditions in order to insure the admissibility of various spaces under the integral operator. Examples are given to illustrate the results.

In chapter 4, we consider the system of integral equations of the form

$$x(t) = f(t) + \int_0^t a(t,s)\ x(s)ds + \int_0^t b(t,s)g(s,x(s))\ du(s) \qquad (3)$$

where $x, f \in BV(J,R^n)$, $g: J \times R^n \to R^n$, $J = [0, \infty)$, $a(t,s)$, $b(t,s)$ are $n \times n$ matrices defined for $0 \leq s \leq t < \infty$, u is a function of bounded variation and right continuous on J and the second integral on the right side of (3) is to be understood as a Stieltjes integral. In section 4.2, we obtain sufficient conditions in order that a certain stability property of the unperturbed system (1) implies a corresponding local stability of (3). Sections 4.3 and 4.4 are devoted to the existence (local as well as global) of solutions of (3) and the asymptotic equivalence between the solutions of (1) and (3). The main tools are the resolvent kernels, Helly's selection principle and fixed point theorems. Number of examples are given to illustrate the results.

In the systems given by ordinary differential equations the values of the state variables change continuously with respect to time. When a physical system contain impulses, there occur discontinuous changes in the state variables of the system. Examples of such systems are pulse frequency modulation systems and models for biological neural nets. The ordinary differential equations are not suitable to deal with the systems containing impulses. Such systems, in fact, give rise to equations of the form

$$Dx = f(t,x) + g(t,x)\ Du \tag{4}$$

where Du denotes the distributional derivative of the function u. If u is a function of bounded variation, Du can be identified with a Stieltjes measure and has the effect of suddenly changing the state of the system at the points where u is discontinuous. For example, if u is the Heaviside function

$$u(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geq 0 \end{cases}$$

then Du is the Dirac measure (known as Dirac δ-function by physicists and Technologists). The equation (4) may be regarded as a perturbed system of the ordinary differential system

$$x' = f(t,x) \tag{5}$$

where $g(t,x)Du$ represents an impulsive perturbation. A natural question arises under what conditions the stability properties of (5) are shared by the solutions of (4). It seems very difficult to get a satisfactory answer to this question. It may be so because differential and integral inequalities play a vital role in the stability theory; but when we consider the stability of solutions of (4), the fact that its solutions are discontinuous renders many of the differential inequalities unapplicable and the integral inequalities are not available for Stieltjes integrals. However, in chapter 5, an attempt has been made to investigate some stability properties of solutions of (4) with the help of generalized integral

inequalities of Gronwall-Bellman type which are developed in chapter 2.

One of the important techniques in the theory of differential equations is the second method of Lyapunov which has been extensively studied in recent years. In chapter 6, we extend Lyapunov's second method and investigate sufficient conditions for eventual uniform asymptotic stability of the trivial solution of (4).

CHAPTER - 1

INTRODUCTION

Since nonlinear differential and integral equations occur in many problems of contemporary Physics, Engineering and Technology the importance of studying such equations needs no explanation. Whenever a quantitative study of a physical system is undertaken, it is usually done by describing the system in mathematical terms. Physical systems of many types and very real practical interest tend increasingly to require the use of nonlinear equations in their mathematical description, in place of much simpler linear equations which have often sufficed in the past. Thus the qualitative theory of nonlinear differential and integral equations is in a continuous process of development. The first considerable serious study of nonlinear differential and integral equations appears in a remarkable paper by A.M. Lyapunov in 1906 on the subject of the stability of a rotating fluid. Nonlinear problems can be approached in different ways. One of them is to express the nonlinear function as a sum of a linear function and a nonlinear function. In this case the nonlinear function can be treated as a perturbation to the linear system. We shall attempt to compare the solutions of these systems with the solutions of the systems obtained by neglecting the nonlinear term. This is a problem which frequently arises in physical examples, where the linear system is solved and its solution is used to describe

approximately the motion governed by the nonlinear system. We shall see that the solutions of the two systems do behave similarly if the nonlinear term is small in a certain sense. This does not settle the question by any means, as in practice, the nonlinear term is frequently not small enough for the applicability of the above approach. To cover a broader class of perturbations a considerable amount of research has been done in recent years, by a number of authors and many interesting results have been accumulated. Most of these authors are concerned with stability of systems when the perturbations are continuous or sufficiently smooth so that the solutions of the systems are continuous. But in nature, all perturbations cannot be expected to behave so well. So the perturbations of the impulsive type are more realistic. In this case, because of impulsive perturbations the problem becomes more complicated and has the effect of suddenly changing the state of the system.

The objective of this thesis is to study the problem of existence, uniqueness and some stability properties of solutions of volterra integral equations with continuous and discontinuous perturbations, and measure differential equations (that is, ordinary differential equations with impulsive perturbations).

1.2 Brief Review

Corduneanu [11], Levin [30], Miller [36,37,38], Nohel [41,42,43], Rama Mohana Rao [49], Strauss [53] among others have studied the existence, uniqueness and asymptotic behavior of solutions of nonlinear volterra integral equations and many interesting

results have been accumulated. In recent years, Corduneanu [12,13] has studied perturbation problems for integral equations by adapting the admissibility theory of Massera and Schaffer [35] . Antosiewicz [2] has obtained some general results which may be used to study the asymptotic behavior of solutions of volterra integral equations. Levin [29,31] , Levin and Nohel [32,33] , Corduneanu [14] , Lakshmikantham and Rama Mohana Rao [26] and Rama Mohana Rao and Tsokos [50] have studied perturbation problems for certain volterra integrodifferential equations.

Quite recently, Grossman [19] has studied the problem of existence and uniqueness of solutions of the nonlinear volterra integral equation

$$x = f + a_1 g_1(x) + a_2 g_2(x) + \dots + a_n g_n(x)$$

where a_i are continuous linear operators mapping a Fréchet space F into itself, g_i are nonlinear operators in that space, and the solutions were sought which lie in a Banach subspace of F having a stronger topology. Equations of this type with one kernel have been studied by many authors; in particular by Miller [36] , Miller, Nohel and Wong [39] , Nohel [44] , Corduneanu [13] and Strauss [53] .

In the first part of this thesis, the idea of admissibility of integral operators is further exploited to study the existence, uniqueness and some stability properties of solutions of system of

integral equations with several kernels and volterra linear equations with continuous and discontinuous perturbations. This study includes some of the results of the above mentioned authors. Examples are given to illustrate the results.

When a system described by an ordinary differential equation $\frac{dx}{dt} = f(t,x)$ is acted upon by perturbation, the perturbed system is generally given by an ordinary differential equation of the form $\frac{dx}{dt} = f(t,x) + G(t,x)$, where the perturbation term $G(t,x)$ is assumed to be continuous or integrable and as such the state of the system changes continuously with respect to time. But, in physical systems one cannot expect the perturbations to be well behaved and it is therefore important to consider the case when the perturbations are impulsive. This will give rise to equations of the form

$$Dx = f(t,x) + G(t,x)\, Du \tag{1.2.1}$$

where Du denotes the distributional derivative of the function u. If u is a function of bounded variation, Du can be identified with a Stieltjes measure and will have the effect of suddenly changing the state of the system at the points of discontinuity of u. Equations of the form (1.2.1) are generally known as measure differential equations. Quite recently, Schmaedeke [51] has studied the problem of existence and uniqueness of solutions of (1.2.1). But the equation (1.2.1) has a more general interpretation as the perturbation of the ordinary differential equation

$$\frac{dx}{dt} = f(t,x) \tag{1.2.2}$$

where $G(t,x)Du$ represents an impulsive perturbation. A natural question arises: under what conditions the stability properties of solutions of (1.2.2) are shared by the solutions of (1.2.1). It seems very difficult to get a satisfactory answer to this question. It may be so because differential and integral inequalities play a very important role in the stability theory; but when we consider the stability of solutions of (1.2.1), the fact that its solutions are discontinuous renders many of the differential inequalities inapplicable while the integral inequalities are not available for Stieltjes integrals. However, an attempt has been made to study the problem of stability of solutions of (1.2.1), in the latter part of this thesis, in two different directions: (i) through generalized integral inequalities of Gronwall-Bellman and Bihari type , and (ii) by Lyapunov's second method. The stability of systems with respect to impulsive perturbations has also been considered by Barbashin [3] and Zabalishchin [59] . Our study includes some of the results of [3][59] and [54] . Examples are constructed to illustrate the results.

1.3 Outline of the thesis.

The present thesis is divided into six chapters. Chapter 2 is auxiliary in character and contains the requiste mathematical equipment which is a part of the theory of ordinary differential equations and volterra integral equations. The results stated in this chapter are used in our subsequent discussion.

Chapter 3 deals with the existence, uniqueness and some stability properties of solutions of system of volterra integral equations with several kernels and different continuous perturbations.

In chapter 4, the problem of existence, uniqueness and some stability properties of solutions of integral equations of volterra type with discontinuous perturbations (i.e. the perturbations involving Stieltjes integrals) is studied.

Chapter 5 is devoted to investigate some stability properties of solutions of ordinary differential systems with respect to impulsive perturbations. The main tools are the generalized integral inequalities of Gronwall-Bellman and Bihari type which are developed in chapter 2.

Finally, in chapter 6, the Lyapunov's second method is extended to investigate sufficient conditions for eventual uniform asymptotic stability of differential equations with impulsive perturbations.

CHAPTER - 2

PRELIMINARIES AND BASIC RESULTS

2.1 Various spaces and norms:

In this section we state the definition of various spaces along with norms which are freely used in the main body of this thesis. It is assumed that the reader is familiar with the notion of Banach spaces, operators on Banach spaces and main theorems in these areas.

Definition 2.2.1.

A Fréchet space is a topological vector space with the following properties:

(a) It is metrizable (in particular, it is Hausdorff).

(b) It is complete.

(c) It is locally convex.

Let F be a Fréchet space, and let d be a metric defined on F. We define $||x|| = d(x,0)$. It can be shown in any Fréchet space, this defines an additively invariant metric. That is, $||\cdot||$ satisfies all the properties of a norm except the condition $||ax|| = |a| \; ||x||$ where a is a constant.

Let $J = [0, \infty)$ and $R^n =$ the n-dimentional Euclidean space. For $x \in R^n$, we define $|x| = \sum_{i=1}^{n} |x_i|$. The Fréchet spaces which we will encounter as examples are the following.

(i) $C(J) = \{x : x$ is continuous on $J \to R^n\}$.

Then $||x||_0 = \sum_{n=1}^{\infty} 2^{-n} \min \{1, \sup_{0 \leq t \leq n} |x(t)|\}$.

(ii) $LL^p(J) = \{x : x$ measurable and $\int_0^T |x(t)|^p \, dt < \infty$ for all $T > 0\}$; $1 \leq p < \infty$.

Then $||x||^p = \sum_{n=1}^{\infty} 2^{-n} \min \{1, \int_0^n |x(t)|^p \, dt\}$.

For any function $x \in L^\infty(J)$, let $||x||_\infty$ be the norm of x considered as an element of $L^\infty(J)$. Similarly, let $|| \cdot ||_p$ denote the norm of elements of $L^p(J)$, $1 \leq p < \infty$. $BC(J)$ = the set of all bounded and continuous functions from J into R^n. If $x \in BC(J)$, define its norm by

$$||x|| = \sup \{|x(t)| : 0 \leq t < \infty\} .$$

We consider a function x on the set of real numbers and taking values in R^n. We consider all possible partitions of an interval [a,b] in the real numbers into a finite number of subintervals: $a=t_0<t_1 \ldots < t_n=b$. The quantity

$$\text{Sup} \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| = V(x, [a,b]),$$

where the upper bound is taken over all possible partitions of the interval [a,b], is called (by definition) the total variation of the function x on the interval [a,b]. We shall say that x is of bounded variation on $[a, \infty)$, if x has bounded variation on any interval [a,t], $a \leq t < \infty$, and the set of total variations $V(x, [a,t])$ is bounded. By definition, we have

$$V(x, [a,\infty)) = \sup_{t \geq a} V(x, [a,t]).$$

$BV(J,R^n) = BV(J)$ = the space of all functions of bounded variation defined on J and taking values in R^n. The norm of $x \varepsilon BV(J)$ is defined by $||x||^* = V(x,J) + |x(0^+)|$, where 0 is the left end point of J. With this norm, $BV(J)$ is a Banach space.

According to a theorem (Helly's selection principle, cf.[52],p. 74) of Helly, if an infinite family $\mathcal{F}$ of functions $\varepsilon BV([a,b])$ is such that all functions of the family and total variation of all functions of the family are uniformly bounded, then there exists a sequence $\{\phi_k\}$ in the family $\mathcal{F}$ which converges at every point of $[a,b]$ to some function $\phi \varepsilon BV([a,b])$; moreover,

$$V(\phi, [a,b]) \leq \underline{\lim}_{k \to \infty} V(\phi_k, [a,b]).$$

2.2 Admissible Operators.

Let F be a Fréchet space. Let B and D be Banach subspaces of F which admit norms $|| \ ||_B$ and $|| \ ||_D$ respectively. Assume that B and D have stronger topology than F in the sense that convergence in B or D implies convergence in F. Let $T:F \to F$ be a linear operator.

Definition 2.2.1.

The pair of spaces (B,D) is said to be admissible with respect to the operator $T:F \to F$ if and only if $TB \subset D$.

The following result is of importance in admissibility theory, whose proof can be found in [25].

Lemma 2.2.1. Let T be a continuous operator from F into itself.

Suppose that B,D are Banach spaces that are stronger than F and the pair (B,D) is admissible with respect to T. Then, T is a continuous operator from B to D.

Remark 2.2.1. Since T is a continuous operator it is also bounded. It then follows that we can find a constant $K > 0$ such that

$$||Tx||_D \leq K \, ||x||_B, \quad x \in B.$$

2.3 Nonlinear integral equations.

Consider the system of volterra integral equations

$$x(t) = f(t) + \int_0^t a(t,s) \; \{x(s)+g(s,x(s))\}ds+\int_0^t b(t,s)h(s,x(s))ds \tag{2.3.1}$$

and the corresponding linear system

$$y(t) = f(t) + \int_0^t a(t,s) \; x(s) \; ds \tag{2.3.2}$$

where x,y,f,g and h are n-vectors and $a(t,s)$, $b(t,s)$ are $n \times n$ matrices. The solutions $x(t)$ and $y(t)$ of (2.3.1) and (2.3.2) respectively can be written as

$$x(t) = y(t)+\int_0^t R_1(t,s)g(s,x(s))ds+\int_0^t R_2(t,s)h(s,x(s))ds \tag{2.3.3}$$

and

$$y(t) = f(t) + \int_0^t R_1(t,s) \; f(s) \; ds \tag{2.3.4}$$

where $R_1(t,s)$ and $R_2(t,s)$ satisfy

$$R_1(t,s) = a(t,s) + \int_s^t R_1(t,\tau) \; a(\tau,s) \; d\tau \tag{2.3.5}$$

and $$R_2(t,s) = b(t,s) + \int_s^t R_1(t,\tau)\, b(\tau,s)\, d\tau . \qquad (2.3.6)$$

Obviously the solution $R_1(t,s)$ of the resolvent system (2.3.5) is the resolvent kernel of $a(t,s)$. In chapter 3 we present, roughly speaking, sufficient conditions in order that a certain stability of the unperturbed system (2.3.2) implies a corresponding local stability of the system (2.3.1). The method that is adapted rests on the use of the resolvent kernel $R_1(t,s)$ for the linear system (2.3.2) and the familiar contraction mapping principle. Our assumptions on the resolvent kernel may be viewed as sufficient conditions in order to insure the admissibility of various spaces under the integral operator $(T\phi)(t) = \int_0^t R_1(t,s)\, \phi(s)\, ds$.

2.4 Integral Inequalities.

The integral inequalities of Gronwall-Reid-Bellman, Bihari [5] and Langenhop [27] play a vital role in studying the stability properties of solutions of differential equations and integral equations. We need the following lemmas in our subsequent discussion which generalize the above inequalities.

Lemma 2.4.1. (Barbashin [3])

Let u and k be scalar non-negative functions defined and integrable on $[0,T]$. Let f be a scalar non-negative function and a function of bounded variation on $[0,T]$. Then, the inequality

$$u(t) \le f(t) + \int_0^t k(s)\, u(s)\, ds, \quad t \in [0,T] \qquad (2.4.1)$$

implies that

$$u(t) \leq f(0) \exp \left(\int_0^t k(s)ds\right) + \int_0^t \exp\left(\int_s^t k(\tau)d\tau\right)df, \quad t \in [0,T], \qquad (2.4.2)$$

where the integral on the right hand side of (2.4.2) is to be understood as a Lebesgue-Stieltjes integral.

Lemma 2.4.2.

Let u, k and g be non-negative integrable functions on $[a,T]$. Assume that $g(u)$ is monotonic increasing in u and M is a positive constant. Then, if the inequality

$$u(t) \leq M + \int_a^t k(s)\, g(u(s))\, ds, \quad t \in [a,T] \qquad (2.4.3)$$

holds, the inequality

$$u(t) \leq W^{-1} \left[W(M) + \int_a^t k(s)\, ds\right], \quad t \in [a,T] \qquad (2.4.4)$$

remains valid as long as $W(M) + \int_a^t k(s)ds$ lies in the domain of definition of W^{-1}, where the function W is defined by the relation

$$W(\eta) = \int_\varepsilon^\eta \frac{d\tau}{g(\tau)}\ ; \quad \varepsilon > 0,\ \eta \geq 0, \qquad (2.4.5)$$

W^{-1} is the inverse mapping of the function W.

Proof:

Denoting the right hand side of (2.4.3) by $v(t)$, we have $u(t) \leq v(t)$. Since g is an increasing function of u and k is

a non-negative function, we have

$$\frac{g(u(t))\ k(t)}{g(v(t))} \leq k(t),$$

But $v'(t) = g(u(t))k(t)$ almost everywhere. Hence by the definition of W, we have $\frac{d}{dt}\ W\,[v(t)] \leq k(t)$ almost everywhere. Integrating between a and t it follows that

$$W(v(t)) \leq W(M) + \int_a^t k(s)\ ds.$$

Since W^{-1} is also increasing, finally, we have

$$u(t) \leq W^{-1}\ [W(M) + \int_a^t k(s)\ ds].$$

The lemma 2.4.2 is proved in [5] under the assumptions that u, g and k are continuous functions but the integrability of u, g and k is sufficient. The lemmas 2.4.1 and 2.4.2 are used in proving some stability properties of solutions of differential systems with impulsive perturbations in chapter 5.

2.5. Distributions.

In this section distributions and distributional derivatives are defined. Let Ω be a subset of R^n. We denote by $C_c^\infty(\Omega)$ the class of all infinitely partially differentiable functions, defined on Ω, which have compact support. $C_c^\infty(\Omega)$ is a normed linear space with addition, scalar multiplication and norm defined by

$$(\phi_1 + \phi_2)\ (x) = \phi_1(x) + \phi_2(x)$$
$$(\alpha\phi)(x) = \alpha\phi\ (x)$$
$$||\phi|| = \sup_{x \in \Omega} |\phi(x)| .$$

Definition 2.5.1.

A continuous linear functional defined on $C_c^\infty(\Omega)$ is called a distribution on Ω .

It follows from Riesz representation theorem that the set of all complex Borel measures on Ω is, by $\mu \longleftrightarrow F_\mu$, in one-one correspondence with the set of all distributions on Ω , where F_μ is the distribution defined by $F_\mu(\phi) = \int_\Omega \phi d\mu$, $(\phi \in C_c^\infty(\Omega))$.

Let a complex function f defined a.e. on Ω be locally integrable on Ω with respect to the Lebesgue measure (that is, for any compact set K of Ω , $\int_K |f(x)|\, dx < \infty$). Then

$$F_f(\phi) = \int_\Omega f(x)\ \phi(x)\ dx,\ \phi \in C_c^\infty(\Omega),$$

defines a distribution F_f on Ω . Two distributions F_{f_1} and F_{f_2} are equal as functionals $(F_{f_1}(\phi) = F_{f_2}(\phi)$ for every $\phi \in C_c^\infty(\Omega))$ if and only if $f_1(x) = f_2(x)$ a.e. (see, [58]).

The derivative of a distribution F with respect to x^i denoted by $D_i F$ or $\frac{\partial F}{\partial x^i}$, is defined by

$$D_i\ F(\phi) = -F\ (\frac{\partial \phi}{\partial x^i}),\ \phi \in C_c^\infty(\Omega),$$

and is also a distribution on Ω . A distribution is infinitely differentiable in the sense of above definition.

Since a locally integrable function f on an open interval I of real line can be identified with the distribution F_f on I, DF_f $(\equiv \frac{dF_f}{dt})$ will be denoted by Df and called the distributional derivative of f to distinguish from the ordinary derivative $f'(\equiv \frac{df}{dt})$. If f is absolutely continuous, then Df is the ordinary derivative f' (which is defined a.e.), f' being considered equivalent to the distribution $F_{f'}$. If f is of bounded variation then Df is the Lebesgue-Stieltjes measure df.

2.6 Measure differential equations.

Let $\mathfrak{M}$ denote the set of all $n \times m$ matrices of real numbers. The norm of a matrix $M = (M^i_j) \in \mathfrak{M}$ will be defined by $|M| = \sum_{i=1}^{n} \sum_{j=1}^{m} |M^i_j|$. Consider the measure differential equation

$$Dx = f(t,x) + G(t,x)\, Du \tag{2.6.1}$$

where f and G are defined on $J \times R^n$ with values in R^n and $\mathfrak{M}$ respectively, and u is a right continuous function $\in BV(J,R^m)$. Let I be an interval with left end point $t_0 \geq 0$.

Definition 2.6.1.

A function $x(t) = x(t,t_0,x_0)$ is said to be a solution of (2.6.1) through (t_0,x_0) on I if x is right continuous $\in BV(I,R^n)$, $x(t_0) = x_0$ and the distributional derivative of x on (t_0,T) for every $T \in I$ satisfies (2.6.1).

The following lemma is used to prove the integral representation of (2.6.1).

Lemma 2.6.1.

If g is a function integrable with respect to μ, and F is a distribution on Ω given by

$$F(\phi) = \int_{\Omega} \phi \, d\mu, \quad \phi \in C_c^{\infty}(\Omega), \tag{2.6.2}$$

then the product gF defined by

$$gF(\phi) = \int_{\Omega} g \, \phi \, d\mu, \quad \phi \in C_c^{\infty}(\Omega) \tag{2.6.3}$$

is also a distribution on Ω.

Proof.

Since $\phi \in C_c^{\infty}(\Omega)$, it is bounded and μ-measurable and g is given to be μ-integrable. Therefore, $g\phi$ is μ-integrable (by Ex.i, P. 184 of [40]). Thus the right hand side of (2.6.3) is meaningful. gF defined by (2.6.3) is obviously a linear functional on $C_c^{\infty}(\Omega)$. Furthermore,

$$(gF)(\phi) \leq \int_{\Omega} |g| \; |\phi| \quad d\,|\mu| \leq ||\phi|| \int_{\Omega} |F| \quad d\,|\mu|$$

where $|\mu|$ denotes the total variation measure of μ. Therefore,

$$||gF|| = \sup \{|gF(\phi)| \, : \, ||\phi|| \leq 1\} \leq \int_{\Omega} |g| \; d\,|\mu| < \infty ,$$

since μ-integrability of g implies $|\mu|$-integrability of $|g|$ (see lemma 18, P. 113, [17]). Thus gF is bounded (hence continuous)

linear functional on $C_c^\infty(\Omega)$, and is, therefore, a distribution on Ω .

Now, consider the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))\, ds + \int_{t_0}^{t} G(s,x(s))\, du(s). \qquad (2.6.4)$$

Theorem 2.6.1.

$x(t)$ is a solution of (2.6.1) through (t_0,x_0) on an interval I, with left end point t_0, if and only if $x(t)$ satisfies (2.6.4) for $t \in I$.

Proof.

Let $x(t)$ satisfy (2.6.4) for $t \in I$. The integral $\int_{t_0}^{t} f(s,x(s))ds$ is absolutely continuous (hence continuous and of bounded variation) function of t on I. The integral $\int_{t_0}^{t} G(s,x(s))du(s)$ is a function of bounded variation on I and the right continuity of u implies that it is also a right continuous function of t. Therefore, $x(t) \in BV(I,R^n)$ and is right continuous. Obviously $x(t_0) = x_0$. Let T be any arbitrary point in I and let F^i be the distribution on (t_0,T) to be identified with the i-th component $x^i(t)$ of $x(t)$. Then

$$F^i(\phi) = \int_{J_1} [x_0^i + \int_{t_0}^{t} f^i(s,x(s))ds + \int_{t_0}^{t} [G(s,x(s))du(s)]^i]\phi(t)dt \qquad (2.6.5)$$

for all $\phi \in C_c^\infty(\Omega)$ where $J_1 = (t_0,T)$. The distributional derivative is

$$DF^i(\phi)=-F(\phi')=-\int_{J_1}[x_0^i+\int_{t_0}^{t} f^i(s,x(s))ds+\int_{t_0}^{t}\{\sum_{j=1}^{m} G_j^i(s,x(s))du^j(s)\}]\phi'(t)dt$$

where $G_j^i(t,x)$ is i, j-th element of $G(t,x)$ and $u^j(t)$ is the j-th component of $u(t)$. Integration by parts yields

$$-\int_{J_1}[x_0^i+\int_{t_0}^{t} f^i(s,x(s))ds]\phi'(t)dt = \int_{J_1}\phi(t)f^i(t,x(t))dt \qquad (2.6.7)$$

since $\phi(t_0) = \phi(T) = 0$. The function $g(t) = \int_{t_0}^{t} G_j^i(s,x(s))du^j(s)$ is right continuous and is of bounded variation on the interval J_1. We have

$$\int_{J_1} g(t)\phi'(t)dt = \int_{(t_0,T]} g(t)\ d\phi(t) - \int_{\{T\}} g(t)\ d\phi(t).$$

But $\int_{\{T\}} g(t)d\,\phi(t) = 0$ since ϕ is continuous; and

$$\int_{(t_0,T]} g(t)\ d\phi(t) = g(T)\phi(T) - g(t_0)\ \phi(t_0) - \int_{(t_0,T]}\phi(t)\ dg(t)$$

$$= -\int_{(t_0,T)}\phi(t)\ dg(t) - \int_{\{T\}}\phi(t)dg(t), (\text{since } \phi(t_0) = \phi(T) = 0),$$

$$= -\int_{J_1}\phi(t)dg(t)-\phi(T)(g(T)-g(T^-)), (\text{cf. [40],Ex.n,P.185}),$$

$$= -\int_{J_1}\phi(t)\ dg(t).$$

Therefore,

$$\int_{J_1} g(t)\ \phi'(t)\ dt = -\int_{J_1}\phi(t)\ dg(t).$$

That is,

$$\int_{J_1}\{\int_{t_0}^{t} G_j^i(s,x(s))du^j(s)\}\phi'(t)dt=-\int_{J_1}\phi(t)d[\int_{t_0}^{t} G_j^i(s,x(s))du^j(s)]$$

$$= - \int_{J_1} \phi(t)\, G^i_j(t,x(t))du^j(t),$$ (cf. [17] corollary 6, P.180).

Summation over j in the above equation yields

$$\int_{J_1} \{ \int_{t_0}^{t} \sum_{j=1}^{m} (G^i_j(s,x(s))\, u^j(s)) \}\phi'(t)dt = -\int_{J_1} \phi(t) [\sum_{j=1}^{m} G^i_j(t,x(t))du^j(t)]$$

From (2.6.6), (2.6.7) and (2.6.8), we obtain

$$DF^i(\phi) = \int_{J_1} \phi(t)f^i(t,x(t))dt + \int_{J_1} \phi(t)[G(t,x(t))du(t)]^i.$$

By lemma 2.6.1, the last continuous linear functional in (2.6.9) is identified with the measure $G(t,x(t))du(t)$. The first continuous linear functional in (2.6.9) is identified with $f(t,x(t))$. Thus the derivative $DF(\phi)$ is identified with $f(t,x(t)) + G(t,x(t))Du$. Hence $x(t)$ is a solution of (2.6.1) through (t_0,x_0).

Conversely, let $x(t)$ be a solution of (2.6.1) through (t_0,x_0) on the interval I. Then for $J_1 = (0,T)$, where T is any arbitrary point of I, we have

$$\int_{J_1} \phi(t)Dx^i(t) = \int_{J_1} \phi(t)f^i(t,x(t))dt + \int_{J_1} \phi(t)[G(t,x(t))du(t)]^i, i=1,2,\ldots,n.$$

for all $\phi \in C_c^\infty(J_1)$. Integrating the left hand side of (2.6.10) by parts and using (2.6.7) and (2.6.8) we obtain

$$\int_{J_1} \phi'(t)(x^i(t)-x^i(t_0))dt = \int_{J_1} \phi'(t)[\int_{t_0}^{t} f^i(s,x(s))ds + \int_{t_0}^{t}[G(s,x(s))du(s)]^i]dt.$$

Therefore,

$$x^i(t) = x^i(t_0) + \int_{t_0}^{t} f^i(s,x(s))ds + \int_{t_0}^{t} [G(s,x(s))du(s)]^i \qquad (2.6.11)$$

almost everywhere in J_1. But, since $x^i(t)$ is right continuous being a solution of (2.6.1), and since the right hand side of (2.6.11) is a function of t that is right continuous, then equality holds everywhere in J_1 for (2.6.11). Thus $x(t)$ satisfies (2.6.4) for $t \in I$. This completes the proof.

CHAPTER - 3

INTEGRAL EQUATIONS OF VOLTERRA TYPE AND ADMISSIBILITY THEORY

3.1 Introduction

It is well known that the theory of admissibility developed by Massera and Schaffer [35] for ordinary linear differential equations plays a vital role in the qualitative study of integral equations of volterra type. It has recently been applied to integral equations by Corduneanu [12,13] , Antosiewicz [2] and others.

The objective of this chapter is to exploit further the idea of admissibility of integral operators to study the problem of existence, uniqueness and some stability properties of solutions of system of integral equations.

Consider the system of volterra integral equations

$$x(t) = f(t) + \int_0^t a(t,s)\, x(s)\, ds \tag{3.1.1}$$

and

$$x(t) = f(t) + \int_0^t a(t,s)[x(s) + g(s,x(s))]\, ds + \int_0^t b(t,s)\, h(s,x(s))\, ds \tag{3.1.2}$$

where x, f, g and h are vectors in R^n and $a(t,s)$, $b(t,s)$ are $n \times n$ matrices. Equation (3.1.2) can be written in the abstract form

$$x = f + Tx + Tg^*(x) + Sh^*(x) \tag{3.1.3}$$

where

$$(Tg^*(x))(t) = \int_0^t a(t,s)\ g(s,x(s))\ ds$$

$$(Sh^*(x))(t) = \int_0^t b(t,s)\ h(s,x(s))\ ds.$$

In the earlier works [2] ,[12,13] , the admissibility of the pair of spaces with respect to continuous linear operators T and S are assumed, but it is not the case in the present study. This amounts to saying that even though the kernels involved in the given integral equations are not well behaved, still one can have the usual existence and uniqueness theorems.

To study the problem of existence, uniqueness and some stability properties of solutions of integral equations, one can have a choice (i) of introducing a linear bounded operator which compensates the behavior of the kernels involved in the integral equations, or (ii) through the behavior of resolvent kernels corresponding to the kernels involved in the integral equations. Each approach has certain advantages. In section 3.2, we adopt the former approach and in section 3.3 the latter. We shall compare in various ways the solution of (3.1.2) with the solution of the (unperturbed) linear system (3.1.1) when g and h have certain "smallness" properties. In section 3.4, the results of sections 3.2 and 3.3 are extended to integral equations with several kernels and different perturbations. Examples are constructed to illustrate the results.

3.2 Operator Equations.

Consider the integral equation

$$x = f + Tx + Tg^*(x) + Sh^*(x) \tag{3.2.1}$$

where x, f are elements of a Fréchet space F, the operators T, S are linear continuous maps from $F \to F$ and g^*, h^* are nonlinear maps from $F \to F$. Solutions are sought which lie in the Banach subspace B of F with a stronger topology.

We need the following conditions for our subsequent discussion.

(H_1) $f \in B$.

(H_2) T and S are continuous linear operators from $F \to F$.

(H_3) There exists an operator ω which belongs to $BL(B,B)$ (the set of bounded linear operators from $B \to B$) and satisfies:

(a) the pair of spaces (B,B) is admissible with respect to linear continuous operators $T + \omega T$ and $S + \omega S$.

(b) $||T + \omega T - \omega||_B < 1$.

(c) $(I + \omega)^{-1}\omega$ is a continuous operator mapping $F \to F$.

(H_4) For each $\eta_1 > 0$, there exists a $\delta_1 > 0$ such that

$$||g^*(x_1) - g^*(x_2)||_B \leq \eta_1 \; ||x_1 - x_2||_B$$

for all $||x_1||_B$, $||x_2||_B \leq \delta_1$, $g^*(0) \equiv 0$ and $g^* : B \to B$.

(H_5) For each $\eta_2 > 0$, there exists a $\delta_2 > 0$ such that

$$||h^*(x_1) - h^*(x_2)||_B \leq \eta_2 \; ||x_1 - x_2||_B$$

for all $||x_1||_B$, $||x_2||_B \leq \delta_2$, $h^*(0) \equiv 0$ and $h^*(x) : B \to B$.

The main result of this section is the following.

Theorem 3.2.1.

Let (H_1) - (H_5) be satisfied. Then, there exist ε_1 and $\varepsilon_2 > 0$ such that if $||f||_B \leq \varepsilon_1$, there is a unique solution x to (3.2.1) which lies in B and in addition $||x|| \leq \varepsilon_2$.

Proof :

Define a vector v by

$$v = -(I + \omega)^{-1}\omega.$$

Then it is easy to show that $(I + v)^{-1} = (I + \omega)$, where I is the identity operator in the Frechet space F. By adding vx to both sides of (3.2.1), we have

$$x + vx = f + vx + Tx + Tg^*(x) + Sh^*(x)$$

Multipling both sides by $(I + v)^{-1}$, we obtain

$$x = \hat{f} + Rx + (T+\omega T)g^*(x) + (S+\omega S)\, h^*(x) \equiv Ux$$

where $\hat{f} = f + \omega f$ which lies in B, $R = T + \omega T - \omega$

To complete the proof of this theorem, we show that U maps $B \to B$ and that U is a contraction. We note that $f \in B$ and $x \in B$ implies that $g^*(x)$ and $h^*(x) \in B$ by (H_4) and (H_5) respectively. By hypothesis (H_3a) and lemma (2.2.1) there exist constants $k > 0$, $L > 0$ such that

$$||(T+\omega T)x||_B \leq k\, ||x||_B \text{ and } ||(S+\omega S)\, x||_B \leq L\, ||x||_B.$$

Also $T + \omega T$, $S + \omega S$ and $\omega \in BL(B,B)$ by (H_3a), so U maps $B \to B$. Now, we let $\lambda = ||R||_B < 1$ by condition (H_3b). Fix $\eta_1, \eta_2 > 0$ such that

$$\eta_1 = (1-\lambda)/4k \text{ and } \eta_2 = (1-\lambda)/4L.$$

Choose δ_1 and $\delta_2 > 0$ so that (H_4) and (H_5) are satisfied. Define $\varepsilon_0 = \min(\delta_1, \delta_2)$. For $0 < \varepsilon_2 \leq \varepsilon_0$, choose

$$\varepsilon_1 = \varepsilon_2 \; (1-\lambda)/2 \; ||I+\omega||_B$$

and define

$$S(\varepsilon_2) = \{x \in B : ||x||_B \leq \varepsilon_2\}.$$

Now, if $x_1, x_2 \in S(\varepsilon_2)$, we have

$$||Ux_1 - Ux_2|| \leq \lambda ||x_1 - x_2|| + k \; ||g^*(x_1) - g^*(x_2)|| + L \; ||h^*(x_1) - h^*(x_2)||$$

$$\leq \lambda \; ||x_1 - x_2|| + k \frac{(1-\lambda)}{4k} ||x_1 - x_2|| + L \frac{(1-\lambda)}{4L} ||x_1 - x_2||$$

$$\leq \frac{(1-\lambda)}{2} ||x_1 - x_2||.$$

Since $\lambda < 1$, U is indeed a contraction on $S(\varepsilon_2)$.

To show that U maps $S(\varepsilon_2) \to S(\varepsilon_2)$, let $x \in S(\varepsilon_2)$ so that

$$||Ux|| \leq ||I+\omega|| \;\; ||f|| + \lambda \; ||x|| + k\eta_1 \; ||x|| + L\eta_2 \; ||x||$$

$$\leq \frac{(1-\lambda)}{2} \varepsilon_2 + \lambda\varepsilon_2 + k\eta_1 \varepsilon_2 + L\eta_2 \varepsilon_2$$

$$= \varepsilon_2.$$

This completes the proof of the theorem.

Example 3.2.1.

Let $F = C([0,\infty),R)$, $B = BC([0,\infty),R)$ where B is a Banach space with the usual supremum norm. Assume that $f, g^*(x)$ and $h^*(x)$ satisfy the conditions stated in theorem 3.2.1. Let the operators T and S be given by the relations

$$(Tx)(t) = \int_0^t (-1 + e^{-\alpha s})\ x(s)\ ds, \ \alpha > 2$$

$$(Sx)(t) = \int_0^t \left[-\frac{t}{(s+2)^3}\right] x(s)\ ds$$

Define the operator ω by the relation

$$(\omega x)(t) = \int_0^t [-e^{-(t-s)}]\ x(s)\ ds.$$

With a little computation, one can show that

$$||T + \omega T||_B \leq 1 + \frac{1}{\alpha-1}.$$

Similarly, it can be shown that $||S + \omega S||_B$ is finite. Thus $T + \omega T$ and $S + \omega S$ map $B \to B$ but T and S do not map $B \to B$. Further

$$||T + \omega T - \omega||_B \leq \frac{1}{\alpha-1} < 1, \text{ since } \alpha > 2$$

Hence all the conditions of theorem 3.2.1 are satisfied.

3.3 Resolvent kernels.

The solutions $y(t)$ and $x(t)$ of (3.1.1) and (3.1.2) respectively can be written as

$$y(t) = f(t) + \int_0^t R_1(t,s)\ f(s)\ ds \qquad (3.3.1)$$

and

$$x(t) = y(t) + \int_0^t R_1(t,s)g(s,x(s))\,ds + \int_0^t R_2(t,s)h(s,x(s))\,ds \tag{3.3.2}$$

(see section 3, chapter 2) where $R_1(t,s)$ and $R_2(t,s)$ satisfy the equations

$$R_1(t,s) = a(t,s) + \int_s^t R_1(t,u)a(u,s)\,du \tag{3.3.3}$$

and

$$R_2(t,s) = b(t,s) + \int_s^t R_1(t,u)\, b(u,s)\, du \tag{3.3.4}$$

Remark 3.3.1.

The equation (3.3.2) can also be considered as a generalization of the control system

$$\begin{aligned} x' &= A(t)\, x + g(t,x) + B(t)\, u \\ x(0) &= x_0 \end{aligned} \tag{3.3.5}$$

where u is a control vector. Indeed, if we denote by $\Phi(t)$ the matrix function defined by $\dot{\Phi} = A(t)\Phi$, $\Phi(0) = I$, then the control system (3.3.5) reduces to

$$x(t)=\Phi(t)x_0+\int_0^t \Phi(t)\Phi^{-1}(s)g(s,x(s))\,ds + \int_0^t \Phi(t)\Phi^{-1}(s)B(s)u(s)\,ds \tag{3.3.6}$$

By setting $R_1(t,s) = \Phi(t)\,\Phi^{-1}(s)$

$$R_2(t,s) = \Phi(t)\,\Phi^{-1}(s)\,B(s)$$

$$y(t) = \Phi(t)\,x_0$$

and $u(t) = h(t,x(t))$,

equation (3.3.6) is equivalent to equation (3.3.2).
In particular, for simplicity of implementation, the one favoured in practice is a linear feedback policy

$$u(t) \equiv Kx(t),$$

wherein $u(t)$ is a fixed linear function of the instantaneous state $x(t)$ given by the feedback operator K. Obviously the system of integral equations (3.1.2) includes the control system (3.3.5) but it is not the case in [19] and [39] .

In the following theorems, our assumptions on the resolvent kernels may be viewed as sufficient conditions in order to insure the admissibility of various spaces under the integral operators

$$(P\phi)(t) = \int_0^t R_1(t,s)\phi(s)\, ds$$

and

$$(Q\phi)(t) = \int_0^t R_2(t,s)\phi(s)\, ds$$

Throughout this section we shall assume that the following hypotheses are satisfied.

(H_1^*) $R_1(t,s)$ and $R_2(t,s)$ are locally integrable in (t,s) for $0 \leq s \leq t < \infty$ and satisfies (3.3.3) and (3.3.4) respectively.

(H_2^*) $g(t,x)$ and $h(t,x)$ are measurable in (t,x) for $0 \leq t < \infty$ and $|x| < \infty$, and $g(t,0) \equiv 0$, $h(t,0) \equiv 0$.

(H_3^*) For each $\eta_1 > 0$, there exists a $\delta_1 > 0$ such that

$$|g(t,x_1) - g(t,x_2)| \leq \eta_1 \, |x_1 - x_2|$$

uniformly in $t \geq 0$, whenever $|x_1|$, $|x_2| \leq \delta_1$.

(H_4^*) For each $\eta_2 > 0$, these exists a $\delta_2 > 0$ such that

$$|h(t,x_1) - h(t,x_2)| \leq \eta_2 \, |x_1 - x_2|$$

uniformly in $t \geq 0$, whenever $|x_1|$, $|x_2| \leq \delta_2$.

Theorem 3.3.1. Let (H_1^*) - (H_4^*) be satisfied. Suppose that the solution $y(t)$ of the linear system (3.1.1) satisfies $y \in L^\infty(J)$. Further, assume that

$$\sup_{t \geq 0} \int_0^t |R_i(t,s)| \, ds \leq c_i \qquad (i = 1,2) \tag{3.3.7}$$

where $c_i (i = 1,2)$ are positive constants.

For any fixed $\lambda \in (0,1)$ there exists a number $\varepsilon_0 > 0$ such that $0 < \varepsilon \leq \varepsilon_0$, if $||y||_\infty \leq \lambda \varepsilon$, then there exists a unique solution $x(t)$ of the system (3.1.2), $x \in L^\infty(J)$ and in addition $||x||_\infty \leq \varepsilon$.

Remark 3.3.2.

Uniqueness here means uniqueness within the class of $L^\infty(J)$ of solutions for which $||x||_\infty \leq \varepsilon$. Note that if $f \in L^\infty(J)$, then assumption (3.3.7) used in (3.3.1) implies that $y \in L^\infty(J)$.

A similar theorem can be stated when $y \in L^\infty(J) \cap L^1(J)$ under some conditions on $R_1(t,s)$ and $R_2(t,s)$.

Theorem 3.3.2.

Suppose that (H_1^*) - (H_4^*) hold. Let $y \in BC(J)$. Assume that $R_1(t,s)$ and $R_2(t,s)$ satisfy

(i) the condition (3.3.7)

(ii) $$\lim_{h \to 0} \left(\int_t^{t+h} |R_i(t+h,s)|\, ds + \int_0^t |R_i(t+h,s)-R_i(t,s)|\, ds \right) = 0, \quad (i=1,2)$$

for each $t \geq 0$.

There exists a number $\varepsilon_0 > 0$ such that $0 < \varepsilon \leq \varepsilon_0$, if $||y|| \leq \lambda\varepsilon$ for some fixed $\lambda \in (0,1)$, then there exists a unique solution $x(t)$ of the equation (3.1.2), $x \in BC(J)$ and in addition $||x|| \leq \varepsilon$.

Corollary 3.3.1.

If the hypotheses of theorem 3.3.2 are satisfied, if for each $T > 0$

$$\lim_{t \to \infty} \int_0^T |R_i(t,s)|\, ds = 0 \quad , (i = 1,2)$$

and if

$$\lim_{t \to \infty} y(t) = 0, \text{ then } \lim_{t \to \infty} x(t) = 0.$$

The proofs of theorems 3.3.1, 3.3.2 and corollary 3.3.1 are similar to the proofs of the corresponding theorems given in [39] and hence omitted.

Remark 3.3.3.

Theorems 3.3.1, 3.3.2 and corollary 3.3.1 may be regarded as stability results for the system (3.1.2)

Example 3.3.1.

Choose $a(t,s) \equiv -1$ and $b(t,s) = \dfrac{(t-s+1)}{(1+s)^2}$. Then, we have $R_1(t,s) = -e^{-(t-s)}$ and $R_2(t,s) = \dfrac{1}{(1+s)^2}$. Obviously $R_1(t,s)$

and $R_2(t,s)$ satisfy the condition (3.3.7) but $a(t,s)$ and $b(t,s)$ fail to satisfy a similar condition. Further, if we define operator ω by the relation

$$(\omega x)(t) = \int_0^t R_1(t,s)x(s)\, ds,$$

one can easily show that $T+\omega T$ and $S+\omega S$ map $B \to B$ but T and S do not map $B \to B$. This verifies the hypothesis (H_3) of section 3.2.

3.4 Extensions.

In this section we shall extend some of the results of sections 3.2 and 3.3 to nonlinear volterra integral equations of the form

$$x(t) = f(t) + \int_0^t a_1(t,s)\, G_1(s,x(s))ds + \int_0^t a_2(t,s)\, G_2(s,x(s))\, ds$$

$$+ \dots + \int_0^t a_n(t,s)G_n(s,x(s))ds + \int_0^t b(t,s)h(s,x(s))\, ds \qquad (3.4.1)$$

Equation (3.4.1) can be written in the abstract form

$$x = f + T_1\, G_1^*(x) + \dots + T_n\, G_n^*(x) + S\, h^*(x) \qquad (3.4.2)$$

where x, f are elements of a Frechet space F, the operators $T_1, T_2 \dots T_n$ and S are linear continuous map from $F \to F$ and $G_1^*,\, G_2^*, \dots, G_n^*$ and h^* are nonlinear maps from $F \to F$.

We need the following conditions:

(C_1) $T_1, T_2, \dots, T_n$ and S are linear continuous operators from $F \to F$.

(C_2) $f \in B$.

(C_3) There exists an operator $\omega \in BL(B,B)$ such that

(a) the pair of spaces (B,B) is admissible with respect to the linear operators $T_i + \omega T_i$ $(i = 1,2,\ldots n)$ and $S + \omega S$.

(b) $\left\|\sum_{i=1}^{n} T_i + \sum_{i=1}^{n} \omega T_i - \omega\right\|_B < 1$

(c) $(I+\omega)^{-1}\omega$ is a continuous operator mapping $F \to F$.

(C_4) $G_i^*(x) = x + g_i^*(x)$ $(i=1,2,\ldots n)$. Further, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|g_i^*(x_1) - g_i^*(x_2)\|_B \leq \varepsilon\, \|x_1 - x_2\|_B, \quad (i = 1,2,\ldots,n.)$$

for all $\|x_1\|_B$, $\|x_2\|_B \leq \delta$, $g_i^*(0) \equiv 0, (i=1,2,\ldots,n)$ and $g_i^* : B \to B$

(C_5) condition (H_5) holds.

Now we state the following general result.

Theorem 3.4.1.

Let (C_1) - (C_5) be satisfied. Then, there exist two positive numbers ε_1 and ε_2 such that if $\|f\|_B \leq \varepsilon_2$, there is a unique solution x to (3.4.2), which lies in B and $\|x\|_B \leq \varepsilon_1$.

The proof of this theorem is similar to that of theorem 3.2.1 and hence omitted.

Remark 3.4.1.

In general, $G_i^*(x)$ need not be of the form $G_i^*(x) = x+g_i^*(x)$ $(i=1,2,\ldots n)$.

In such cases the condition (C_3b) of theorem 3.4.1 will be replaced by $||\omega||_B < 1$

Remark 3.4.2.

For simplicity, consider the system of equations (3.4.1) when $n = 2$ and $G_i(t,x) = x+g_i(t,x)$ $(i=1,2)$. Then the solution $x(t)$ of (3.4.1) and the solution $y(t)$ of the corresponding linear system can be written as

$$y(t) = f(t) + \int_0^t \{R_1(t,s) + R_1^*(t,s)\} f(s)\, ds$$

and

$$x(t) = y(t) + \int_0^t R_1(t,s)\, g_1(s,x(s))\, ds + \int_0^t R_1^*(t,s) g_2(s,x(s))\, ds$$
$$+ \int_0^t R_2(t,s)\, h(s,x(s))\, ds \qquad (3.4.3)$$

where $R_1(t,s)$, $R_1^*(t,s)$ and $R_2(t,s)$ are given by

$$R_1(t,s) = a_1(t,s) + \int_s^t \{R_1(t,u) + R_1^*(t,u)\}\, a_1(u,s)\, du$$

$$R_1^*(t,s) = a_2(t,s) + \int_s^t R_1(t,u) + R_1^*(t,u)\}\, a_2(u,s)\, du$$

and

$$R_2(t,s) = b(t,s) + \int_s^t \{R_1(t,u) + R_1^*(t,u)\}\, b(u,s)\, du.$$

The results similar to theorems 3.3.1, 3.3.2 and corollary 3.3.1 can be obtained to equation (3.4.3).

CHAPTER - 4

VOLTERRA INTEGRAL EQUATIONS WITH DISCONTINUOUS PERTURBATIONS

4.1 Introduction.

In many problems of Physics and Engineering (optimal control theory in particular) one can not expect the perturbations to be well behaved and it is therefore important to consider the cases when the perturbations are impulsive (cf. [15] , [51]). Such systems would be described by differential equations containing measures which are equivalent to volterra integral equations with perturbations involving Lebesgue-Stieltjes integrals.

Miller, Nohel and Wong [39] , Strauss [53] among others have studied the integral equations of volterra type with continuous perturbations. The aim of this chapter is to study the existence, uniqueness and some stability properties of solutions of integral equations of volterra type with discontinuous perturbations.

Consider the system of integral equations of the form

$$x(t) = f(t) + \int_0^t a(t,s)\, x(s)\, ds \tag{4.1.1}$$

and $$x(t) = f(t) + \int_0^t a(t,s)x(s)ds + \int_0^t b(t,s)g(s,x(s))du(s) \tag{4.1.2}$$

where $x, f \in BV(J)$, $g\colon J \times R^n \to R^n$, $J = [0,\infty)$, $a(t,s)$, $b(t,s)$ are

$n \times n$ matrices defined for $0 \leq s \leq t < \infty$ and u is a function of bounded variation, right continuous on J.

We shall compare in various ways the solution of (4.1.2) with the solution of the (unperturbed) linear system (4.1.1). In section 4.2, we obtain sufficient conditions in order that a certain stability of the unperturbed system (4.1.1) implies a corresponding local stability of the system (4.1.2). Sections 4.3 and 4.4 are devoted to the study of existence (local as well as global) of solutions of (4.1.2) and the asymptotic equivalence between the solutions of (4.1.1) and (4.1.2). Examples are constructed to illustrate the results.

4.2 Existence, Uniqueness and Stability.

The solutions $y(t)$ and $x(t)$ of (4.1.1) and (4.1.2) respectively, can be written as

$$y(t) = f(t) + \int_0^t R_1(t,s)\, f(s)\, ds \quad (t \geq 0) \tag{4.2.1}$$

and

$$x(t) = y(t) + \int_0^t R\,(t,s)\, g(s,x(s))\, du(s) \quad (t \geq 0), \tag{4.2.2}$$

where $R_1(t,s)$ and $R(t,s)$ satisfy

$$R_1(t,s) = a(t,s) + \int_s^t R_1(t,\tau)\, a(\tau,s)\, d\tau \tag{4.2.3}$$

and

$$R(t,s) = b(t,s) + \int_s^t R_1(t,\tau)\, b(\tau,s)\, d\tau \tag{4.2.4}$$

Obviously the solution $R_1(t,s)$ of the resolvent system (4.2.3) is the resolvent kernel of $a(t,s)$.

Remark 4.2.1.

If $a(t,s) \equiv b(t,s)$, then $R_1(t,s) \equiv R_2(t,s)$. Then the system (4.1.2) reduces to the system considered in [39,51] when u is an absolutely continuous function on J.

Remark 4.2.2.

The system (4.2.2) can also be considered as a generalization of the system of differential equations

$$\frac{dy}{dt} = A(t)\, y \tag{4.2.5}$$

and

$$Dx = A(t)\, x + g(t,x)Du \tag{4.2.6}$$

where $x,y \in R^n$, $A(t)$ is a $n \times n$ matrix, Du denotes the distributional derivative of the function u. $g : J \times R^n \to R^n$ and $u : J \to R$ is a function of bounded variation and right continuous on J. Here Du can be identified with a Stieltjes measure and has the effect of suddenly changing the state of the system at the points of discontinuity of u.

Indeed, if we denote by $X(t)$ the matrix function defined by $\dot{X}(t) = A(t)\, X(t)$, $X(0) = I$, then the solution $y(t)$ of (4.2.5) can be written as

$$y(t) = X(t)\, y_0, \; y(0) = y_0$$

Suppose that $x(t)$ be any solution of (4.2.6) and let

$$x(t) = X(t)\, c(t), \; c(0) = y_o \tag{4.2.7}$$

where $X \varepsilon C^{\infty}(J)$.

By taking the distributional derivative on both sides of (4.2.7), we obtain (cf. [18])

$$Dc = X^{-1} g(t,x)\, Du$$

which is equivalent to

$$c(t) = c(0) + \int_0^t X^{-1}(s)\; g(s)x,s))\; du(s).$$

Thus in view of (4.2.7), we have

$$x(t) = y(t) + \int_0^t X(t)\, X^{-1}(s)\; g(s,x(s))\; du(s). \tag{4.2.8}$$

By setting $R(t,s) = X(t)\, X^{-1}(s)$, system (4.2.8) is similar to system (4.2.2).

We require the following conditions in our subsequent discussion.

($\bar{H}_1$) u is a right continuous real valued function defined on J and is a function of bounded variation on J.

($\bar{H}_2$) $R(t,s)$ is integrable with respect to u for fixed t, $0 \leq s \leq t < \infty$. Let Φ denote a finite set of ordered numbers

$$\Phi : 0 = t_o < t_1 < \ldots < t_N = t$$

and

$$\psi(t) = \sup_{\Phi} \left[\sum_{i=1}^{N} \left\{ \int_{t_0}^{t_{i-1}} |R(t_i,s)-R(t_{i-1},s)|\,dV(s) + \right.\right.$$

$$\left.\left. + \int_{t_{i-1}}^{t_i} |R(t_i,s)|\, dV(s) \right\}\right]$$

where $|du(s)| = dV(s)$.

$(\bar{H}_3)$ $\sup_{t \geq 0} [\psi(t)] \leq B_0$, where B_0 is a positive constant.

The main result of this section is the following.

Theorem 4.2.1.

Let $(\bar{H}_1)$-$(\bar{H}_3)$ be satisfied. Assume that $g(t,x)$ satisfies

(i) $g(t,x)$ is integrable (in the sense of Lebesgue-Stieltjes) with respect to u and $g(t,0) \equiv 0$.

(ii) For each $L > 0$ there exists $\delta > 0$ such that

$$|g(t,x) - g(t,y)| \leq L\, |x-y|$$

Uniformly in $t \geq 0$ whenever $|x|$, $|y| \leq \delta$.

Suppose that $y \in BV(J)$ ($y(t)$ is the solution of the linear system (4.1.1)). For any fixed $\lambda \in (0,1)$ there exists a number $\varepsilon_0 > 0$ such that $0 < \varepsilon_1 \leq \varepsilon_0$, if $||y||^* \leq \lambda\, \varepsilon_1$ then there exists a unique solution $x(t)$ of the system (4.1.2), $x \in BV(J)$ and in addition $||x||^* \leq \varepsilon_1$.

Proof.

For a fixed $\lambda \in (0,1)$, select $L > 0$ such that $LB_0 \leq (1-\lambda)$. By assumption (ii) of the theorem, we have a $\delta > 0$ such that

$$|g(t,x_1) - g(t,x_2)| \leq L\,|x_1-x_2|$$

uniformly in $t \geq 0$, whenever $|x_1|, |x_2| \leq \delta$. Now, let $\varepsilon_o = \delta$ and for any $\varepsilon_1 > 0$, $0 < \varepsilon_1 \leq \varepsilon_o$, define the set

$$S(\varepsilon_1) = \{x \in BV(J) : ||x||^* \leq \varepsilon_1\}$$

and an operator A by the relation

$$(Az)(t) = y(t) + \int_0^t R(t,s)\; g(s,z(s))\; du(s)$$

for $z \in S(\varepsilon_1)$. Thus, we have

$$||Az||^* \leq ||y||^* + ||\int_0^t R(t,s)g(s,z(s))\; du(s)||^*$$

$$\leq \lambda\,\varepsilon_1 + \sup_{t \geq 0}\; \sup_{\Phi}\; [\sum_{i=1}^{N} \{\int_{t_o}^{t_{i-1}} |R(t_i,s)-R(t_{i-1},s)|L|z(s)|\; dV(s)$$

$$+ \int_{t_{i-1}}^{t_i} |R(t_i,s)|\; L|z(s)| \quad dV(s)\}]$$

Therefore, by using $(\bar{H}_2)$ and $(\bar{H}_3)$, we obtain

$$||Az||^* \leq \lambda\,\varepsilon_1 + L\,B_o\,\varepsilon_1 \leq \lambda\,\varepsilon_1 + (1-\lambda)\,\varepsilon_1 = \varepsilon_1.$$

Hence A maps $S(\varepsilon_1)$ into itself.

Now, if $z_1, z_2 \in S(\varepsilon_1)$, by using $(\bar{H}_2)$ and $(\bar{H}_3)$ and the assumption (ii) of the theorem, we have

$$||Az_1-Az_2||^* \leq L\,B_o\;||z_1-z_2||^*$$

$$\leq (1-\lambda)\;||z_1-z_2||^*$$

Since $\lambda < 1$, A is indeed a contraction on $S(\varepsilon_1)$.

Therefore, the system (4.2.2) and hence the system (4.1.2) has a unique solution $x \in BV(J), ||x||^* \leq \varepsilon_1$. This completes the proof of the theorem.

Remark 4.2.3

A sufficient condition for $||y||^* \leq \lambda\, \varepsilon_1$ is that $||f||^* \leq \lambda\, \varepsilon_1(1+c_o)^{-1}$ where c_o is a positive constant such that

$$\sup_{t \geq 0}\ \sup_{\Phi}\Big[\sum_{i=1}^{N}\Big\{\int_{t_o}^{t_{i-1}}|R_1(t_i,s)-R_1(t_{i-1},s)|\,ds+\int_{t_{i-1}}^{t_i}|R_1(t_i,s)|\,ds\Big\}\Big] \leq c_o < \infty. \tag{4.2.9}$$

Thus theorem 4.2.1 may be regarded as a stability result for the system (4.1.2) in the following sense: Given any $\varepsilon_1 > 0$ there exists a $\varepsilon_2 > 0$ such that for every $f \in BV(J)$, $||f||^* \leq \varepsilon_2$ implies that the solution $x(t)$ of (4.1.2) is in $BV(J)$ and $||x||^* \leq \varepsilon_1$.

Remark 4.2.4.

It is apt to remark here that the hypothesis $(\bar{H}_3)$ implies that

$$\sup_{t \geq 0} V\Big\{\int_0^t |R(t,s)|\, dV(s),\ [0,t]\Big\} \leq B_o \tag{4.2.10}$$

and the condition (4.2.9) implies

$$\sup_{t \geq 0} V\Big(\int_0^t |R_1(t,s)|\, ds,\ [0,t]\Big) \leq c_o. \tag{4.2.11}$$

The following corollary is an immediate consequence of theorem 4.2.1.

Corollary 4.2.1.

If the hypotheses of theorem 4.2.1 are satisfied and in addition

$(\bar{H}_4)$: for each $T > 0$, $\lim\limits_{t \to \infty} \int_0^T |R(t,s)| \, dV(s) = 0$

then $\lim\limits_{t \to \infty} y(t) = 0$ implies that $\lim\limits_{t \to \infty} x(t) = 0$.

Proof.

Define the set

$$S_o(\varepsilon_1) = \{x \in S(\varepsilon_1) : \lim_{t \to \infty} x(t) = 0\}$$

clearly $S_o(\varepsilon_1)$ is a closed subspace of $S(\varepsilon_1)$ under the norm $||\cdot||^*$. We need only to prove that $(Ax)(t) \to 0$ as $t \to \infty$ for any $x \in S_o(\varepsilon_1)$. Let $\eta > 0$ be given. By hypotheses choose $T > 0$ such that

$$|x(t)| \leq \frac{\eta}{3}, \; |y(t)| \leq \frac{\eta}{3} \text{ for all } t \geq T.$$

Choose $T_1 > T$ so large that, by (H_4)

$$\int_0^T |R(t,s)| \, dV(s) \leq \frac{\eta B_o}{3\varepsilon_1} \; (t \geq T_1).$$

Then for $t \geq T_1$, we have

$$|(Ax)(t)| \leq |y(t)| + \int_0^t |R(t,s)| \; |g(s,x(s))| dV(s)$$

$$\leq \eta/3 + \int_0^T |R(t,s)|\ L\ |x(s)|\ dV(s) + \int_T^t |R(t,s)|\ L\ |x(s)|\ d\,V(s)$$

$$\leq \eta/3 + L\ \varepsilon_1 \int_0^T |R(t,s)|\ dV(s)$$

$$+ \frac{\eta L}{3} \int_T^t |R(t,s)|\ d\,V(s)$$

$$\leq \eta/3 + L\ \varepsilon_1 \frac{\eta\ B_o}{3\ \varepsilon_1} + \frac{\eta L}{3} \int_0^t |R(t,s)|\ dV(s)$$

$$\leq \eta/3 + \frac{L\,\eta\,B_o}{3} + \frac{L\ \eta\ B_o}{3}$$

$< \eta$, since $L\ B_o \leq 1-\lambda$ by the choice of L in theorem 4.2.1.

Since $\eta > 0$ is arbitrary, $(Ax)(t) \to 0$ as $t \to \infty$. Thus $AS_o(\varepsilon_1) \subset S_o(\varepsilon_1)$ and one completes the proof in the usual way.

Example 4.2.1 (when $n = 1$).

Choose $a(t,s) = e^{e^{-t}-e^{-s}-s}$. Then we have $R_1(t,s) = e^{-s}$ for $0 \leq s \leq t < \infty$. Obviously the condition (4.2.9) is satisfied. Further, let

$$f(t) = \begin{cases} \beta\ e^{-t} & \text{for } t < \alpha \\ \beta\ e^{-2t} & \text{for } t \geq \alpha \end{cases}$$

where α and β are positive constants.

Then, one can verify that the solution $y(t)$ of the linear system (4.2.1) is

$$y(t) = \begin{cases} \beta e^{-t} + (\beta/2)\ (1-e^{-2t}) & \text{for } t < \alpha \\ \beta e^{-2t} + (\beta/2)\ (1-e^{-2\alpha}) + (\beta/3)\ [e^{-3\alpha} - e^{-3t}] & \text{for } t \geq \alpha. \end{cases}$$

which is a function of bounded variation on J.

Clearly $||y||^* \leq \lambda\, \varepsilon_1$ by a proper choice of β. Now, choose

$$u(t) = \begin{cases} 0 & \text{for } t < r \\ 1 & \text{for } t \geq r \end{cases}, \quad r > 0$$

and $b(t,s) = \dfrac{e^{e^{-t}-e^{-s}}}{(1+s)^2}$ for $0 \leq s < t < \infty$.

Then we have $R(t,s) = \dfrac{1}{(1+s)^2}$ and we see that all the conditions of Theorem 4.2.1 are satisfied.

Example 4.2.2 (when n = 1).

Select

$$u(t) = \begin{cases} 0 & \text{for } t < r \\ (1+t)^3\ e^{-t} & \text{for } t \geq r \end{cases}, \quad r > 0.$$

and $b(t,s) = e^{e^{-t}-e^{-s}}$ for $0 \leq s \leq t < \infty$. Then it is clear that the hypotheses $(\bar{H}_2)$ and $(\bar{H}_3)$ hold. By selecting f as in Example 4.2.1, we see that all the conditions of Theorem 4.2.1 are satisfied.

Example 4.2.3 (when n = 1)

Choose $a(t,s) \equiv -1$. Then, we have $R_1(t,s) = -e^{-(t-s)}$. Clearly, the condition (4.2.11) is satisfied but not the condition (4.2.9).

However, the condition (4.2.9) implies the condition (4.2.11).

4.3 Method of Successive Approximations.

In the section, we shall give an existence theorem by using the method of successive approximations.

Let S be a domain (an open connected set) in R^n. The set of all functions in $BV([0,T])$, $T > 0$ with values in S will be denoted by $BV([0,T], S)$. For each $t \in [0,T]$, we shall denote by Q_t the set of all functions x with properties

(i) $x \in BV([0,t])$

(ii) $V(x, [0,t]) \leq b$ where $b > 0$.

Suppose that $Q_t \subset BV([0,t], S)$. This is always possible if b is suitably chosen.

Theorem 4.3.1.

Assume that

(i) the hypothesis $(\bar{H}_1)$ holds,

(ii) the solution $y(t)$ of the linear system (4.1.1) exists, right continuous and $y \in BV([0,T], S)$,

(iii) for each $x \in BV([0,T], S)$, $g(t,x)$ is integrable (in Lebesgue-Stieltjes sense) with respect to the Lebesgue-Stieltjes measure du on $[0,T]$,

(iv) $\lim_{k \to \infty} x_k(t) = x^*(t)$ for each $t \in [0,T]$ and $\lim_{k \to \infty} t_k = t^*$ imply
$\lim_{k \to \infty} g(t_k, x_k(t)) = g(t^*, x^*(t^*))$
where $t_k \in [0,T]$.

(v) there exists a real positive continuous function w defined over $[0,T]$ such that

$$|g(t,x)| \leq w(t), \; t \in [0,T]$$

uniformly with respect to $x \in BV([0,T],S)$,

and

(vi) the function $\psi(t)$ defined in hypothesis $(\bar{H}_2)$ is sufficiently small for small t.

Then, there exists a solution of (4.1.2) on some interval $[0,a]$, $a > 0$.

Proof.

Let $\lambda \in (0,1)$ be fixed. Then, by hypotheses (ii) and (iv) choose a, $0 < a \leq T$ such that

$$V(y,[0,a]) \leq \lambda \, b \text{ and } \psi(a) \leq \frac{(1-\lambda)}{M} b$$

where $M = \sup\limits_{0 \leq t \leq a} w(t)$.

Now, we define the sequence of functions $\{x_k\}$ by setting

$$x_k(t) = \begin{cases} y(t) & \text{for } t \in [0,a/k] \\ y(t) + \int_{a/k}^{t} R(t,s)g(s-a/k, x_k(s-a/k))du(s) & \\ \qquad \text{for } t \in [\frac{ja}{k}, \frac{(j+1)}{k} a], \; j = 1,2,\dots,k-1 . & \end{cases} \tag{4.3.1}$$

The first equation in (4.3.1) defines $x_k(t)$ on the interval $[0,a/k]$; the second equation in (4.3.1) defines $x_k(t)$ at first on the interval $[a/k, 2a/k]$, then on the interval $[2a/k, 3a/k]$, and so on. Thus x_k is defined on the interval $[0,a]$. Therefore

we have

$$V(x_k,[0,a]) \leq V(y,[0,a]) + V(\int_{a/k}^{t} R(t,s)g(s-a/k,\ x_k(s-a/k))\ du(s),[0,a])$$

$$\leq \lambda\ b + \sup_{\Phi_1} [\sum_{i=1}^{N} \{\int_{t_0'}^{t_{i-1}'} |R(t_i',s) - R(t_{i-1}',s)|\ dV(s)$$

$$+ \int_{t_{i-1}'}^{t_i'} |R(t_1',s)|\ \ dV(s)\}]$$

where $\qquad \Phi_1 : a/k = t_o' < t_1' < \dots < t_N' = t.$

Hence,

$$V(x_k,[0,a]) \leq \lambda\ b + M\ \psi(a) \leq b.$$

Therefore, $x_k \in Q_a$ all $k = 1,2,\dots$.

This implies that the sequence $\{x_k\}$ is uniformly bounded and is of uniformly bounded variation. By Helly's selection principle (see [52]) there exists a subsequence $\{x_{k_j}\}$ and a function x^* of bounded variation such that

$$\lim_{j \to \infty} x_{k_j}(t) = x^*(t),\ t \in [0,a]$$

and $\quad V(\ x^*,\ [0,a]\ ) \leq b.$

This shows that $x^* \in Q_a \subset BV(\ [0,a]\ ,S).$

From equations (4.3.1), we have

$$x_{k_j}(t) = \begin{cases} y(t) & \text{for } t \in [0,a/k_j] \\ y(t)+\int_0^t R(t,s)g(s-a/k_j,x_{k_j}(s-a/k_j))du(s) & \\ \quad -\int_0^{a/k_j} R(t,s)g(s-a/k_j,x_{k_j}(s-a/k_j))du(s) & \text{for } t \in [a/k_j,a]. \end{cases}$$

Let du^+ and du^- be the positive and negative variations of Lebesgue-Stieltjes measure du. Then

$$\int_0^t R(t,s)\, g(s-a/k_j,\, x_{k_j}(s-a/k_j))\, du(s)$$

$$= \int_0^t R(t,s) g(s-a/k_j, x_{k_j}(s-a/k_j))\, du^+(s)$$

$$- \int_0^t R(t,s)\, g(s-a/k_j,\, x_{k_j}(s-a/k_j))\, du^-(s)$$

Now take limits on both sides as $j \to \infty$. By using the hypothesis (iv) and Lebesgue's dominated convergence theorem, we obtain

$$\lim_{j \to \infty} \int_0^t R(t,s)\, g(s-a/k_j\, ,\, x_{k_j}(s-a/k_j))\, du(s)$$

$$= \int_0^t R(t,s)\, g(s,x^*(s))\, du^+(s) - \int_0^t R(t,s) g(s,x^*(s))\, du^-(s)$$

$$= \int_0^t R(t,s)\, g(s,x^*(s))\, du(s)$$

and

$$\lim_{j \to \infty} \int_0^{a/k_j} R(t,s)\, g(s-a/k_j,\, x_{k_j}(s-a/k_j))\, du(s) = 0\ .$$

Thus, from (4.3.2) as $j \to \infty$, we have

$$x^*(t) = y(t) + \int_0^t R(t,s)g(s,x^*(s))\, du(s),\ t \in [0,a].$$

This completes the proof of the theorem.

4.4. Global Existence Theorem and Asymptotic Equivalence.

The main disadvantage of proving the existence theorem using the method of successive approximations is that a continuation theorem is necessary. But one can avoid the continuation theorem (which is difficult in our case because of discontinuous solutions) by proving the existence of solution of (4.1.2) on the whole interval J. This could be achieved by using fixed point theorems. Thus, we have the following existence theorem on J.

Theorem 4.4.2.

Assume that

(i) the hypotheses $(\bar{H}_1)$ - $(\bar{H}_3)$ hold,

(ii) the solution $y(t)$ of the linear system (4.1.1) exists and $y \in BV(J)$,

(iii) $x \to g(t,x)$ is a continuous map mapping $BV(J)$ into $BV(J)$,

(iv) $x_n \to x$ pointwise as $n \to \infty$ implies $g(t,x_n(t)) \to g(t,x(t))$ as $n \to \infty$ uniformly in $t \in J$,

(v) there exists a non-negative measurable function $W(t)$ and a positive number ρ such that

$$|g(t,x)| \leq W(t) \quad \text{for } |x| \leq \rho,\ t \in J$$

and $$||y||^* + A_0 \leq \rho$$

where A_0 is a positive constant such that

$$\sup_{t \geq 0} \sup_{\Phi} \Big[\sum_{i=1}^{N} \Big\{ \int_{t_0}^{t_{i-1}} |R(t_i,s) - R(t_{i-1},s)|\, W(s)\, dV(s) + \int_{t_{i-1}}^{t_i} |R(t_i,s)|\, W(s)\, dV(s) \Big\}\Big] \leq A_0.$$

Then there exists at least one solution $x(t)$ of the system (4.1.2) on J and in addition $||x||^* \leq \rho$.

Proof.

Define a set

$$S(\rho) = \{x \in BV(J) : ||x||^* \leq \rho\}$$

and an operator A by the relation

$$(Ax)(t) = y(t) + \int_0^t R(t,s)\ g(s,x(s))\ du(s)$$

for $x \in S(\rho)$.

From our hypothesis it follows immediately that A is a continuous operator mapping $S(\rho)$ to $BV(J)$. Further it can be shown that

$$AS(\rho) \subset S(\rho).$$

Indeed, for every $x \in S(\rho)$, we have

$$||Ax||^* \leq ||y||^* + ||\int_0^t R(t,s)\ g(s,x(s))\ du(s)||^*$$

$$\leq ||y||^* + \sup_{t \geq 0}\ \sup_{\Phi_1} [\sum_{i=1}^{N} \{\int_{t_0}^{t_{i-1}} |R(t_i,s)|\ W(s)\ dV(s)$$

$$\int_{t_{i-1}}^{t_i} |R(t_i,s)|\ dV(s)\}].$$

Thus by using the assumption (v) of the theorem, we obtain

$$||Ax||^* \leq ||y||^* + A_0 \leq \rho.$$

We now show that $AS(\rho)$ is a relatively compact set in $BV(J)$.

It suffices to prove that, any sequence $\{Ax_n\}_{n=1}$ in $AS(\rho)$ has a convergent subsequence. Obviously, the sequence $\{x_n\}$ is uniformly bounded and is of uniformly bounded variation. Hence by Helly's selection principle (cf. [52] , p. 74) there exists a subsequence $\{x_{n_j}\}$ which converges pointwise to $x^* \varepsilon S(\rho)$. However, this does not imply that $||x_{n_j} - x^*||^* \to \infty$ as $j \to \infty$. Therefore, it remains to prove that $||Ax_{n_j} - Ax^*||^* \to 0$ as $j \to \infty$. In fact

$$||Ax_{n_j} - Ax^*||^* = ||\int_0^t R(t,s)\ [g(s,x_{n_j}(s)) - g(s,x^*(s))]\ du(s)||^*$$

$$\leq \sup_{t \geq 0}\ \sup_{\Phi} [\sum_{i=1}^{N} \{\int_{t_0}^{t_{i-1}} |R(t_i,s) - R(t_{i-1},s)|$$

$$|g(s,x_{n_j}(s)) - g(s,x^*(s))|\ dV(s)$$

$$+ \int_{t_{i-1}}^{t_i} |P(t_i,s)||g(s,x_{n_j}(s)) - g(s,x^*(s))|\ dV(s).$$

Let $\varepsilon > 0$ be given. Choose $N_1 > 0$ so large that, by assumption (iv) of the hypotheses,

$$|g(s,x_{n_j}(s)) - g(s,x^*(s))| < \varepsilon/B_0 \text{ for } n_j > N_1.$$

Then, we have

$$||Ax_{n_j} - Ax^*||^* < \varepsilon \text{ for } n_j > N_1$$

and hence $||Ax_{n_j} - Ax^*||^* \to 0$ as $j \to \infty$

Now, an application of Schander's fixed point theorem yields the desired result. This completes the proof of the theorem.

Corollary 4.4.1.

Let the assumptions of Theorem 4.4.1 hold and in addition $(\bar{H}_4)$ holds. Assume that $W(t)$ is bounded and $\lim_{t \to \infty} W(t) = 0$. Then

$$|x(t) - y(t)| \to 0 \text{ as } t \to \infty.$$

The proof of this corollary is direct and hence omitted.

Remark 4.4.1. In theorem 4.4.1, the assumption (v) can be replaced by the following assumption.

(v)' there exists a nonnegative bounded measurable function $W(t)$ and a positive number ρ such that

$$|g(t,x)| \leq W(t) \text{ for } |x| \leq \rho,\ t \in J$$

and

$$||y||^* + M_1 B_0 \leq \rho,$$

where
$$M_1 = \sup_{t \in J} W(t).$$

CHAPTER - 5

ON THE STABILITY OF DIFFERENTIAL SYSTEMS WITH RESPECT TO IMPULSIVE PERTURBATIONS

5.1 Introduction.

It is well known that the integral inequalities of Gronwall-Reid-Bellman type [4] and its extensions ([5] , [27]) play a vital role in studying the qualitative behavior of solutions of ordinary differential equations and volterra integral equations. But they are not much useful to study the behavior of solutions of measure differential equations and differential equations with impulsive perturbations. The reason for this is, the integral inequalities of Gronwall-Reid-Bellman type and its extensions are not available for the Stieltjes integrals. However, a slight modification of Gronwall-Reid-Bellman inequality and its extensions would be useful to obtain some stability properties of solutions of ordinary differential equations containing measures.

The object of this chapter is to investigate some stability properties of solutions of ordinary differential systems with respect to impulsive perturbations. Sections 5.2 and 5.3 contain theorems on uniform asymptotic stability of zero solution of a system of ordinary differential equations containing measures.

5.2 Stability.

We shall consider the following measure differential equation

$$Dx = f(t,x) + g(t,x)\, Du \tag{5.2.1}$$

where $x \in R^n$, Du denotes the distributional derivative of the function u. $f,g \in C[J \times R^n, R^n]$ and $u: J \to R$ is a function of bounded variation, right continuous on J. Here Du can be identified with a Stieltjes measure and will have the effect of suddenly changing the state of the system at the points of discontinuity of u. The primary aim of considering (5.2.1) is the following. The equation (5.2.1) may be regarded as a perturbed system of the ordinary differential equation

$$x' = f(t,x) \tag{5.2.2}$$

where the perturbation $g(t,x)Du$ is impulsive and has the effect of suddenly changing the state of the system. A natural question arises under what conditions the stability properties of (5.2.2) are shared by the solutions (5.2.1). It seems very difficult to get a satisfactory answer to this question. It may be so because differential and integral inequalities play a very important role in the stability theory [25] ; but when we consider the stability of solutions of (5.2.1), the fact that its solutions are discontinuous renders many of the differential inequalities unapplicable while the integral inequalities are not available for Stieltjes integrals. However, a generalization of Gronwall-Reid-Bellman inequality and its

extension, (given in chapter 2) would be useful to investigate some stability properties of solutions of (5.2.1) with respect to the solutions of (5.2.2). Let f and g be smooth for local existence and uniqueness [cf. 46] .

Let $x(t) = x(t,t_0,x_0)$ and $y(t) = y(t,t_0,x_0)$ be the solutions of (5.2.2) and (5.2.1) respectively, through (t_0,x_0), existing to the right of $t_0 \geq 0$ in $S(r)$ where $S(r) = \{x \in R^n: |x| \leq r\}$. The function $y(t,t_0,x_0)$ is a solution of (5.2.1) with $y(t_0) = x_0$ if and only if

$$y(t) = x_0 + \int_{t_0}^{t} f(s,y(s))ds + \int_{t_0}^{t} g(s,y(s))\ du(s).$$

Definition 5.2.1.

The null solution of (5.2.1) is said to be eventually uniformly asymptotically stable if the following two conditions hold:

(i) for every $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and $\tau = \tau(\varepsilon) > 0$ such that $|y(t,t_0,x_0)| < \varepsilon$, $t \geq t_0 \geq \tau(\varepsilon)$, provided $|x_0| < \delta$,

(ii) for every $\eta > 0$, there exist positive numbers δ_0 , τ_0 and $T = T(\eta)$ such that

$$|y(t,t_0,x_0)| < \eta,\ t \geq t_0 + T,\ t_0 \geq \tau_0,$$

provided $|x_0| < \delta_0$.

In this section, we shall give sufficient conditions for uniform asymptotic stability of (5.2.1) with respect to (5.2.2). Let $f(t,0) \equiv 0$ for all $t \geq 0$.

Theorem 5.2.1.

Let the null solution of (5.2.2) be uniformly asymptotically stable. Assume that

(i) f satisfies a Lipschitz condition of the type $|f(t,x)-f(t,y)| \leq L(t)\,|x-y|$, for $|x|, |y| \leq a$ where a is a positive real number and $L(t)$ is a non-negative real-valued integrable function satisfying

$$\sup_{t \geq 0} \int_t^{t+\tau} L(s)\, ds \leq A(\tau),$$

where $A(\tau) > 0$ is a continuous function for all $\tau > 0$,

(ii) there exists $r > 0$ such that if $|x| \leq r$, then $|g(t,x)| \leq \gamma(t)$ for all $t \geq 0$, where

$$G(t) = \int_t^{t+1} \gamma(s)\, dV(s) \to 0 \text{ as } t \to \infty.$$

Then, there exist $T_0 \geq 0$ and $\delta_0 > 0$ such that if $t_0 \geq T_0$ and $|x_0| < \delta_0$, the solution $y(t,t_0,x_0)$ of (5.2.1) satisfies $|y(t,t_0,x_0)| \to 0$ as $t \to \infty$. In particular, if $g(t,0) \equiv 0$, then the null solution of (5.2.1) is eventually uniformly asymptotically stable.

Proof.

Constants corresponding to the system (5.2.2) shall be starred, those for (5.2.1) shall not. Without loss of generality we can assume $r \leq a \leq \delta_0^*$. Let $t_0 \geq 1$ and $|x_0| \leq r$. Then if $|y(t,t_0,x_0)| \leq r$ on $[t_0,t_0+\tau]$ for some $\tau > 0$, we have

$$|y(t) - x(t)| \leq \int_{t_0}^{t} L(s)\ |y(s) - x(s)|\ ds + \int_{t_0}^{t} \gamma(s)\ dv(s).$$

obviously, the set of points M where the function $\int_{t_0}^{t} \gamma(s)\ dv(s)$ is discontinuous, is atmost countable and its Lebesgue measure is zero. On the other hand, the function $\int_{t_0}^{t} \gamma(s)\ dv(s)$ is continuous on $[t_0, \infty) - M$, therefore it is integrable on $[t_0, \infty)$. Hence, by Lemma 2.3.1, we get

$$\begin{aligned}|y(t) - x(t)| &\leq \int_{t_0}^{t} \exp\left[\int_{s}^{t} L(\tau)\ d\tau\right] \gamma(s)\ dv(s)\\ &\leq \exp\left[A(\tau)\right] \int_{t_0}^{t} \gamma(s)\ dv(s).\end{aligned}$$

Since $t \geq t_0 \geq 1$, by changing the order of integration, we obtain

$$|y(t) - x(t)| \leq \exp\left[A(\tau)\right] \int_{t_0 - 1}^{t} G(s)\ ds.$$

Define $Q(t) = \sup\ \{G(s) : t-1 \leq s < \infty\}$.

Then $Q(t)$ decreases to zero as $t \to \infty$ and

$$|y(t) - x(t)| \leq (1+\tau) \exp\left[A(\tau)\right] Q(t_0).$$

Let $0 < \varepsilon \leq r$. Choose $\delta = \delta(\varepsilon) = \delta^*(\frac{\varepsilon}{2})$ so that $0 < \delta < \varepsilon$. Choose $\tau = \tau(\varepsilon) = T^*(\frac{\delta}{2})$. Choose $T_1 = T_1(\varepsilon) \geq 1$ so large that

$$Q(T_1) < \delta\ \left[2(1+\tau) \exp\{A(\tau)\}\right]^{-1} .$$

Let $t_0 \geq T_1$ and $|x_0| < \delta$. Then for as long as $|y(t, t_0, x_0| \leq r$

in the interval $[t_0, t_0+\tau]$,

$$|y(t,t_0,x_0)| \leq |y(t,t_0,x_0) - x(t,t_0,x_0)| + |x(t,t_0,x_0)|$$
$$\leq \exp [A(\tau)] (1+\tau) Q(T_1) + \frac{\varepsilon}{2}$$
$$< \frac{\delta}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

Let $y_1 = y(t_0+\tau, t_0, x_0)$ Then

$$|y_1| \leq |y_1 - x(t_0+\tau, t_0, x_0)| + |x(t_0+\tau, t_0, x_0)|$$
$$< \frac{\delta}{2} + x(t_0 + T^*(\tfrac{\delta}{2}), t_0, x_0)|$$
$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Now, let m be a positive integer and assume that

$|y(t,t_0,x_0)| < \varepsilon$ on $[t_0, t_0+m\tau]$ and $|y(t_0+m\tau, x_0, t_0)| < \delta$. Let $y_m = y(t_0 + m\tau, t_0, x_0)$. Then for as long as $|y(t, t_0+m\tau, y_m)| \leq r$ on the interval $[t_0+m\tau, t_0+(m+1)\tau]$, we have

$$|y(t,t_0+m\tau,y_m)| \leq |y(t,t_0+m\tau,y_m) - x(t,t_0+m\tau,y_m)| + |x(t,t_0+m\tau,y_m)|$$

$$< \exp [A(\tau)] (1+\tau) Q(t_0+m\tau) + \frac{\varepsilon}{2}$$

$$< \frac{\delta}{2} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Let $y_{m+1} = y(t_0+(m+1)\tau, t_0, x_0)$. Then

$$|y_{m+1}| \leq |y(t_0+(m+1)\tau, t_0+m\tau, y_m) - x(t_0+(m+1)\tau, t_0+m\tau, y_m)| +$$
$$+ |x(t_0+(m+1)\tau, t_0+m\tau, y_m)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Thus, by induction, $|y(t,t_0,x_0)| < \varepsilon$ on every interval $[t_0 + m\tau, t_0 + (m+1)\tau]$, and hence on $[t_0, \infty)$. Hence, if $g(t,0) \equiv 0$

then we have shown that the null solution of (5.2.1) is eventually uniformly stable. For the rest of the proof, choose $\delta_o = \delta(r)$ and $T_o = T_1(r)$. Fix $t_o \geq T_o$ and $|x_o| < \delta_o$. Then $|y(t,t_o,x_o)| < r$ on $[t_o, \infty)$. Let $0 < \eta < r$. Choose $\delta(\eta) = \delta^*(\frac{\eta}{2})$, $0 < \delta < \eta$, $\tau(\eta) = T^*(\frac{\delta}{2})$ and $T_1(\eta)$ such that

$$Q(T_1) < \delta[2(1+\tau) \exp\{A(\tau)\}]^{-1}.$$

Let $y_o = y(t_o + T_1, t_o, x_o)$. Then $|y_o| < r \leq \delta_o^*$. Then

$$\begin{aligned} |y(t_o+\tau+T_1,t_o+T_1,y_o)| &\leq |y(t_o+\tau+T_1,t_o+T_1,y_o)-x(t_o+\tau+T_1,t_o+T_1,y_o)| + \\ &\quad + |x(t_o+\tau+T_1,t_o+T_1,y_o)| \\ &\leq \exp[A(\tau)]\,(1+\tau)\,Q(t_o+T_1) + \frac{\delta}{2} \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Thus by the first part of the proof,

$$|y(t,t_o,x_o)| = |y(t,t_o+\tau+T_1,y(t_o+\tau+T_1))| < \eta \text{ for all } t \geq t_o + T,$$

where $T = T(\eta) = \tau + T_1$, completing the proof.

Theorem 5.2.2.

Assume that all the conditions of Theorem 5.2.1 are satisfied except that condition (i) is replaced by

$$|f(t,x) - f(t,y)| \leq \lambda(t)\, W(|x-y|)$$

for $|x|$, $|y| \leq r$, where $W(0) = 0$ and $W(u)$ is monotonic increasing in u, $\lambda(t)$ is a non-negative integrable function such that

$\int_0^\infty \lambda(s)ds$ is finite.

Then there exist $T_o \geq 0$ and $\delta_o > 0$ such that if $t_o \geq T_o$ and $|x_o| < \delta_o$, the solution $y(t,t_o,x_o)$ of (5.2.1) satisfies $|y(t,t_o,x_o)| \to 0$ as $t \to \infty$ provided that $\int_0^1 \frac{dt}{W(t)}$ is divergent. If $g(t,0) \equiv 0$, then the null solution of (5.2.1) is eventually uniformly asymptotically stable.

Proof.

As in the proof of Theorem 5.2.1, let $t_o \geq 1$ and $|x_o| \leq r$. Then if $|y(t,t_o,x_o)| \leq r$ on $[t_o,t_o+\tau]$ for some positive number τ we have

$$\begin{aligned}|y(t,t_o,x_o)-x(t,t_o,x_o)| &\leq \int_{t_o}^{t} \lambda(s)\; W(|y(s)-x(s)|)ds + \int_{t_o}^{t} \gamma(s)\; dv(s)\\ &\leq \int_{t_o}^{t} \lambda(s)\; W(|y(s)-x(s)|)ds + Q(t_o)\; (1+\tau)\end{aligned}$$

where Q is the function defined in Theorem 5.2.1. Thus, by applying Lemma 2.4.2 on $[t_o,t_o+\tau]$, we have

$$|y(t,t_o,x_o) - x(t,t_o,x_o)| \leq \Omega^{-1}\; [\Omega\{(1+\tau)\; Q(t_o)\} + \int_{t_o}^{t} \lambda(s)\; ds]$$

where $\Omega(u) = \int_{u_o}^{u} \frac{ds}{W(s)}$, $u_o > 0$, $u \geq 0$.

Since $\int_0^1 \frac{ds}{W(s)}$ is divergent (or $\Omega(0) = -\infty$), for a given $\varepsilon_1 > 0$ there exists a positive number $K_1 = K_1(\varepsilon_1)$, however large, such that

$$\Omega^{-1}(s) \leq \varepsilon_1 \text{ whenever } s \leq -K_1(\varepsilon_1)$$

and also for a given positive number K_2 there exists a positive number $\varepsilon_2 = \varepsilon_2(K_2)$, however small, such that

$$\Omega(s) \leq -K_2 \text{ for } s \leq \varepsilon_2(K_2).$$

Let $0 < \varepsilon \leq r$ be given, choose $\delta = \delta(\varepsilon) = \delta^*(\frac{\varepsilon}{2})$ so that $0 < \delta < \varepsilon$. Choose $\tau = \tau(\varepsilon) = T^*(\frac{\delta}{2})$. Let $\varepsilon_1 = \frac{\delta}{2}$ and define $K_2 = K_1(\frac{\delta}{2}) + \int_0^\infty \lambda(s)ds$. Choose $T_1 \geq 1$ so large that

$$Q(T_1) < \frac{\varepsilon_2(K_2)}{(1+\tau)}.$$

Let $t_0 \geq T_1$ and $|x_0| < \delta$. Then as long as $|y(t,t_0,x_0)| \leq r$ for $t \in [t_0, t_0+\tau]$, we have

$$\begin{aligned}
|y(t,t_0,x_0)| &\leq |y(t) - x(t)| + |x(t,t_0,x_0)| \\
&\leq \Omega^{-1}\left[\Omega\{\varepsilon_2(K_2)\} + \int_0^\infty \lambda(s)ds\right] + \frac{\varepsilon}{2} \\
&\leq \Omega^{-1}\left[-K_1(\tfrac{\delta}{2})\right] + \frac{\varepsilon}{2} \\
&< \frac{\delta}{2} + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Thus on $[t_0, t_0+\tau]$ we have shown that $|y(t,t_0,x_0)| < \varepsilon$ whenever $|x_0| < \delta$ and $t_0 \geq T_1$. Let $y_1 = y(t_0+\tau, t_0, x_0)$, then it is easy to show that $|y_1| < \delta$. The rest of the proof is exactly in Theorem 5.2.1 and hence omitted.

5.3. Special cases.

In this section we shall consider the linear differential systems with impulsive perturbations and present some results which

generalize some of the results of [3] , [6] , [7] and [54] . To this end, we assume that

$$f(t,x) = Ax + F(t,x)$$

in equation (5.2.1) where A is a $n \times n$ constant matrix and $F \in C[J \times R^n, R^n]$. The solution $y(t)$ of

$$Dx = Ax + F(t,x) + g(t,x)Du \tag{5.3.1}$$

satisfying $y(t_o) = x_o$, $t_o \in J$, is given by

$$y(t)=\Phi(t)x_o+\int_{t_o}^{t}\Phi(t-s)F(s,y(s))ds+\int_{t_o}^{t}\Phi(t-s)g(s,y(s))du(s), t \in J, \tag{5.3.2}$$

where $\Phi(t)$ is the fundamental matrix of the equation $x' = Ax$ satisfying $\Phi(t_o) = I$ and $\Phi \in C^{\infty}(J)$ [cf. 46] .

Theorem 5.3.1.

Assume that

(i) the characteristic roots of A have negative real parts;

(ii) given any $\varepsilon > 0$, there exist $\delta(\varepsilon), T(\varepsilon) > 0$ such that

$$|F(t,x)| \leq \varepsilon\, |x| \text{ provided } |x| < \delta(\varepsilon) \text{ and } t \geq T(\varepsilon) ,$$

and

(iii) the condition (ii) of Theorem 5.2.1 holds.

Then, there exist $T_o \geq 0$ and $\delta_o > 0$ such that for every $t_o \geq T_o$ and $|x_o| < \delta_o$, any solution $y(t) = y(t,t_o,x_o)$ of (5.3.1) satisfies $\lim_{t \to \infty} y(t) = 0$. If, in particular, (5.3.1) possesses the

If $t \geq t_o \geq 1$, by using condition (ii) of Theorem 5.2.1 and changing the order of integration, we obtain

$$\int_{t_o}^{t} c_1 e^{-a(t-s)} \gamma(s)\ dv(s) \leq c_1 e^{-at} \int_{t_o-1}^{t} e^{a(s+1)} G(s)ds$$

$$\leq c_1 Q(t_o)\ e^{-at} \int_{t_o-1}^{t} e^{a(s+1)}\ ds$$

$$\leq \frac{c_1}{a} Q(t_o)\ e^{a}.$$

Therefore, from (5.3.4) for $t \geq t_o \geq 1$, we have

$$|y(t)| \leq c_1\ |x_o| + c_2 Q(t_o) + \int_{t_o}^{t_1} c_1\ \lambda(s)\ w(|y(s)|)ds \quad (5.3.5)$$

where $c_2 = \frac{c_1}{a} e^{a}$.

Hence, by Lemma 2,4.2., we get

$$|y(t)| \leq \Omega^{-1}\ [\Omega(c_1\ |x_o| + c_2\ Q(t_o)) + c_1 \int_{t_o}^{t} \lambda(s)ds] \quad (5.3.6)$$

where $\Omega(u) = \int_{u_o}^{u} \frac{dt}{W(t)}$, $u \geq 0$, $u_o > 0$.

Now, we choose $|x_o|$ and $\Omega(t_o)$ small enough so that $\Omega\ [c_1|x_o| + c_2\ Q(t_o)]$ will be as small as we wanted (it will approach $-\infty$ arbitrarily). Thus the right hand side of (5.3.6) can be made arbitrarily small for all $t \geq t_o$. This proves the eventual uniform stability of the null solution of (5.3.1).

Further, we show that $|y(t)| \to 0$ as $t \to \infty$, provided that $|x_o|$ is sufficiently small. Since the null solution of (5.3.1) is eventually uniformly stable, $|y(t)|$ is bounded for sufficiently small $|x_o|$.

That is,

$$|y(t,t_o,x_o)| < K \quad \text{for all} \quad t \geq t_o$$

whenever $|x_o| < \delta$, where K depends on δ .

Using (5.3.2), (5.3.3) and the assumptions of the theorem, we get

$$|y(t)| \leq c_1 |x_o| e^{-a(t-t_o)} + c_1 \int_0^{t/2} e^{-a(t-s)} \lambda(s) w(K) \, ds + c_1 \int_{t/2}^{t} e^{-a(t-s)} \lambda(s) w(K) \, ds$$

$$+ c_1 \int_{t_o}^{t} e^{-a(t-s)} \gamma(s) \, dv(s)$$

$$\leq c_1 |x_o| e^{-a(t-t_o)} + c_1 w(K) e^{-\frac{at}{2}} \int_0^{t/2} \lambda(s) \, ds + c_1 W(k) \int_{t/2}^{t} \lambda(s) ds$$

$$+ \int_{t_o-1}^{t} c_1 e^{-a(t-s)} G(s) \, ds$$

By using L'Hospital rule and the fact that $G(t) \to 0$ as $t \to \infty$, it is clear that

$$\int_{t_o-1}^{t} e^{-a(t-s)} G(s) \, ds \to 0 \text{ as } t \to \infty.$$

Further, being $\int_0^{\infty} \lambda(s) ds$ is finite, the right hand side of the above inequality tends to zero as $t \to \infty$. Consequently $|y(t)| \to 0$ as $t \to \infty$, which completes the proof of the theorem.

CHAPTER - 6

DIFFERENTIAL SYSTEMS WITH IMPULSIVE PERTURBATIONS AND EXTENSION OF LYAPUNOV'S METHOD

6.1 Introduction.

It is well known that the second method of Lyapunov and its extensions play a vital role in studying the qualitative behavior of solutions of ordinary differential equations (cf. [1], [28], [34]) and functional differential equations (cf. [20], [57]). This method crucially depends on estimating the derivative of a scalar function along the solutions of the given differential equation. But this type of technique is not much useful to study the behavior of solutions of measure differential equations and differential equations with impulsive perturbations. The reason for this is, since the solutions of such differential equations are discontinuous, most of the differential inequalities are not applicable. However, the stability of systems with respect to impulsive perturbations has been considered by Barbashin [3] and Zabalischin [59]. The aim of this chapter is to extend the Lyapunov's second method and investigate sufficient conditions for uniform asymptotic stability of differential equations with impulsive perturbations. The present study includes some of the results of [3], [54] and [59]. A simple example is constructed to illustrate the results.

6.2 Lyapunov Functions

As is well known, Lyapunov's second method has its origin in three simple theorems that form the core of what he himself called the second method for dealing with questions of stability. It is widely recognized as an indispensible tool not only in the theory of stability but also in studying many other qualitative properties of solutions of differential equations. The main characteristic of this method is the construction of a scalar function, namely the Lyapunov function. We need the following definitions.

Definition 2.6.1

A function $\phi(r)$ is said to belong to the class $\mathcal{K}$ if $\phi \in C[[0,\rho),R^+]$, $\phi(0)=0$, and $\phi(r)$ is strictly monotonic increasing in r.

Definition 2.6.2.

A function $V(t,x)$ with $V(t,0) \equiv 0$ is said to be positive definite (negative definite) if there exists a function $\phi(r) \in \mathcal{K}$ such that the relation

$$V(t,x) \geq \phi(|x|) \quad (\leq -\phi(|x|))$$

is satisfied for $(t,x) \in J \times S_\rho$.

Definition 2.6.3.

A function $V(t,x) \geq 0$ is said to be decrescent if a function $\phi(r) \in \mathcal{K}$ exists such that

$$V(t,x) \leq \phi(|x|),\ (t,x) \in J \times S_\rho.$$

Definition 2.6.4.

$\bar{C}_0$ denotes the class of functions having uniform Lipschitz constants on $J \times S_\rho$; C_m the class of functions having continuous partial derivatives of order $k = 1,2,\ldots m$; C_∞ the class of functions having continuous and bounded derivatives of every order on $J \times S_\rho$.

Consider the differential system of the form

$$x' = f(t,x) \tag{6.2.1}$$

where $x,\ f \in R^n$ and $f(t,0) = 0$ for all $t \geq 0$.

Definition 2.6.5.

$V(t,x)$ is a Lyapunov function for (6.2.1) if

(i) $V(t,x)$ is positive definite and C_1 on $J \times S_\rho$,

(ii) $V(t,0) \equiv 0$,

(iii) $\dot{V}_{(6.2.1)}(t,x) = \frac{\partial}{\partial t} V(t,x) + \nabla V(t,x) \cdot f(t,x) \leq 0$ for $(t,x) \in J \times S_\rho$

where $\nabla V = (\frac{\partial}{\partial x_1} V, \ldots, \frac{\partial}{\partial x_n} V)$.

We need the following result whose proof can be found in [34]

Theorem 2.6.1 (Massera [34]).

If, for (6.2.1), $f \in \bar{c}_0$ on $J \times S_\rho$ and $x \equiv 0$ is uniformly asymptotically stable then there exists on $J \times S_\rho$ a Lyapunov function $V(t,x)$ for (6.2.1) such that $V(t,x)$ is positive definite and decrescent and $\dot{V}_{(6.2.1)}$ is negative definite.

6.3 Uniform Asymptotic Stability.

We shall consider the following measure differential equation

$$Dx = f(t,x) + g(t,x)Du \qquad (6.3.1)$$

where $x \in R^n$, Du denotes the distributional derivative of the function u; f,g: $J \times R^n \to R^n$ and u: $J \to R$ is a function of bounded variation, right continuous on J. The primary aim of considering (6.3.1) is that the equation (6.3.1) may be regarded as a perturbed system of the ordinary differential system

$$x' = f(t,x) \qquad (6.3.2)$$

where the perturbation $g(t,x)Du$ is impulsive and has the effect of suddenly changing the state of the system.

The proof of our main result is crucially depending on almost everywhere differentiability of the function u and this property is guranteed because u is a function of bounded variation. In fact, a function of bounded variation has a finite differential coefficient almost everywhere (see [55], P. 356).

We require the following conditions in our subsequent discussion:

(A_1) There exists a $r > 0$ such that for every b, $0 < b < r$, there exist $\tau_b \geq 0$ and a measurable function $\gamma_b(t)$ defined on $[\tau_b, \infty)$ such that $|g(t,x)| \leq \gamma_b(t)$ for all $b \leq |x| \leq r$ and $t \geq \tau_b$, where

$$G_b(t) = \int_t^{t+1} \gamma_b(s)\, ds \to 0 \text{ as } t \to \infty.$$

Define $Q_b(t) = \sup \{G_b(s) : t-1 \leq s < \infty\}$

obviously, $Q_b(t) \downarrow 0$ as $t \to \infty$.

(A_2) u is a right continuous real valued function defined on J and is a function of bounded variation on J. The discontinuties $t_1 < t_2 < t_3 \ldots < t_k < \ldots$ of u are isolated.

(A_3) The series $\sum_{n=1}^{\infty} \gamma_b(t_n)\, h_n$ and $\sum_{n=1}^{\infty} Q_b(t_n)$ are convergent, where h_n represents the jump at t_n of the function u.

(A_4) The derivative of u on $[t_k, t_{k+1})$, $k = 1,2,3,\ldots$ exists and is bounded by a constant $M > 0$ where the derivative at t_k, $k = 1,2,3\ldots$ is to be understood as the right hand derivative.

In this section, we shall give sufficient conditions for eventual uniform asymptotic stability (see for definition [25], P. 222) of the trivial solution of (6.3.1).

Theorem 6.3.1.

Assume that the conditions (A_1) to (A_4) hold and $f \in \bar{C}_0$, $f(t,0) = 0$ for $t \geq 0$. Let the trivial solution of (6.3.2) be uniformly asymptotically stable. Then there exist $T_0 \geq 0$ and $\delta_0 > 0$ such that if $t_0 \geq T_0$ and $|x_0| < \delta_0$, then the solution $y(t,t_0,x_0)$ of (6.3.1) satisfies $|y(t,t_0,x_0)| \to 0$ as $t \to \infty$. In particular, if $g(t,0) \equiv 0$, then the trivial solution of (6.3.1) is eventually uniformly asymptotically stable.

Proof.

Since the trivial solution of (6.3.2) is uniformly asymptotically stable, by Theorem 6.2.1, there exists a Lyapunov function $V(t,x)$ on $J \times S_r$ satisfying

(i) $a(|x|) \leq V(t,x) \leq c(|x|)$, $a, c \in \mathcal{K}$

(ii) $\dot{V}_{(6.3.2)}(t,x) \leq -\sigma(|x|)$, $\sigma \in \mathcal{K}$

(iii) $|\nabla V(t,x)| \leq K_1$, where K_1 is a positive constant.

Further, since u is continuous and differentiable on $[t_k,t_{k+1})$, y is also continuous and differentiable on $[t_k,t_{k+1})$, $k = 1,2,\ldots$ Let $t_k \geq \tau_{b+1}$ be chosen for a fixed k.

Now, as long as $y(t)$ of (6.3.1) exists and differentiable for $t \in [t_k,t_{k+1})$, we have

$$\frac{d}{dt} V(t,y(t)) = \frac{\partial}{\partial t} V(t,y(t)) + \nabla V(t,y(t)).[f(t,y(t))+g(t,y(t))\hat{u}(t)].$$

Integrating from t_k to t and using (i), (ii) and (iii), we get

$$V(t,y(t)) \leq V(t_k,y(t_k)) - \int_{t_k}^{t} \sigma(|y(s)|)\, ds + MK_1 \int_{t_0}^{t} |g(s,y(s))|\, ds.$$

Thus, if $0 < b \leq |y(s)| < r$ between t_k and t, where $t \in [t_k, t_{k+1})$, then the Lemma 3.4 in [54] yields

$$V(t,y(t)) \leq V(t_k,y(t_k)) - \sigma(b)\,(t-t_k) + MK_1\, Q_b(t_k)(t-t_k+1). \qquad (6.3.3)$$

Since $V(t,x)$ is continuous in t for each fixed x, we have

$$V(t_{k+1},y(t_{k+1})) \leq |V(t_{k+1},y(t_{k+1})) - V(t_{k+1}^-,y(t_{k+1}^-))| + V(t_{k+1}^-,y(t_{k+1}^-))$$

$$\leq |V(t_{k+1},y(t_{k+1})) - V(t_{k+1},y(t_{k+1}^-))| + \lim_{h \to 0} [V(t_{k+1}-h, y(t_{k+1}-h))], \qquad (6.3.4)$$

where $h > 0$ is sufficiently small. Further, we know (cf. Theorem 2.6.1) that y is a solution of (6.3.1) through (t_0,x_0) on J if and only if it satisfies the integral equation

$$y(t) = x_0 + \int_{t_0}^{t} f(s,y(s))\, ds + \int_{t_0}^{t} g(s,y(s))\, du(s)$$

for $t \geq t_0 \in J$. Now, by definition

$$|y(t_{k+1}) - y(t_{k+1}^-)| = \lim_{h \to 0} |y(t_{k+1}) - y(t_{k+1}-h)|$$

$$= \lim_{h \to 0} \left| \int_{t_{k+1}-h}^{t_{k+1}} f(s,y(s))\, ds + \int_{t_{k+1}-h}^{t_{k+1}} g(s,y(s))\, du(s) \right|$$

$$\leq |g(t_{k+1}, y(t_{k+1}))|\; |u(t_{k+1}) - u(t_{k+1}^-)|.$$

Thus

$$|y(t_{k+1}) - y(t_{k+1}^-)| \leq \gamma_b(t_{k+1})\, h_{k+1}. \tag{6.3.5}$$

By using (6.3.3), (6.3.4), (6.3.5) and the uniform Lipschitz property of V (this property is guranteed because of (iii)), we obtain

$$V(t_{k+1},y(t_{k+1})) \leq V(t_k, y(t_k)) + \lim_{h \to 0} [-\sigma(b)\ (t_{k+1}-t_k-h)$$

$$+ MK_1\, Q_b(t_k)(t_{k+1}-t_k-h+1)] + K_1\, \gamma_b(t_{k+1})h_{k+1}.$$

That is,

$$V(t_{k+1},y(t_{k+1})) \leq V(t_k,y(t_k)) - \sigma(b)\ (t_{k+1} - t_k) + \tag{6.3.6}$$

$$+ MK_1\, Q_b(t_k)\ (t_{k+1}-t_k+1) + K_1\, \gamma_b(t_{k+1})\, h_{k+1}.$$

Thus, if $0 < b \leq |y(s)| < r$ between t_k and t_{k+1}, then for any $t \in [t_k, t_{k+1}]$ from (6.3.3) and (6.3.6), we have

$$V(t,y(t)) \leq V(t_k,y(t_k)) - \sigma(b)(t-t_k) + MK_1 Q_b(t_k)(t-t_k+1) + K_1\gamma_b(t_{k+1})h_{k+1}. \tag{6.3.7}$$

Similarly, if $b \leq |y(s)| < r$ between t_{k+1} and t_{k+2}, then for $t \in [t_{k+1},t_{k+2}]$, we have

$$V(t,y(t)) \leq V(t_{k+1},y(t_{k+1})) - \sigma(b)(t-t_{k+1}) + MK_1 Q_b(t_{k+1})(t-t_{k+1}+1)$$

$$+ K_1\, \gamma_b(t_{k+1})h_{k+2} \tag{6.3.8}$$

Thus, if $b \leq |y(s)| < r$ between t_k and t_{k+2}, by (6.3.7), (6.3.8) and decreasing property of $Q_b(t)$, we have for $t \in [t_k, t_{k+2}]$

$$V(t,y(t)) \leq V(t_k,y(t_k)) - \sigma(b)\ (t-t_k) + MK_1Q_b(t_k)(t-t_k+1) + MK_1Q_b(t_{k+1})$$

$$+ K_1\ \gamma_b(t_{k+1})\ h_{k+1} + K_1\ \gamma_b(t_{k+2})\ h_{k+2}.$$

By repeating this process, in general, if $b \leq |y(s)| < r$ between t_k and t, then we have

$$V(t,y(t)) \leq V(t_k,y(t_k)) - \sigma(b)(t-t_k) + MK_1\ Q_b\ (t-t_k+1)$$

$$+ \sum_{i=K}^{\infty} K_1[\gamma_b(t_{1+i})\ h_{1+i} + M\ Q_b(t_{i+1})]. \tag{6.3.9}$$

Let $0 < \varepsilon \leq r$ be given. Choose $\delta = \delta(\varepsilon)$, $0 < \delta < \varepsilon$ so that

$$3c(\delta) < a(\varepsilon). \tag{6.3.10}$$

Let $\ell \geq 2$ be sufficiently large fixed number. Choose $N_1 = N_1(\varepsilon)$ so large, $t_{N_1} \geq \tau_{\delta/\ell} + 1$ so that

$$\sum_{n=N_1}^{\infty} K_1[\gamma_{\delta/\ell}\ (t_{n+1})\ h_{n+1} + M\ Q_{\delta/\ell}\ (t_{n+1})] < \frac{a(\varepsilon)}{3} \tag{6.3.11}$$

Choose $T_1 = T_1(\varepsilon) \geq t_{N_1}$ so large that

$$3MK_1\ Q_{\delta/\ell}\ (T_1) < \min\ (\sigma(\delta/\ell),\ a(\varepsilon)). \tag{6.3.12}$$

Let $|x_0| < \delta$ and $t_0 \geq T_1$. Then we claim that

$$|y(t,t_0,x_0)| < \varepsilon \text{ for } t \in [t_0,\infty). \tag{6.3.13}$$

Suppose not. Let T_3 be the first point such that $|y(t_3)| \geq \varepsilon$ and let $T_2 < T_3$ be the last point such that $\delta/\ell \leq |y(T_2)| \leq \delta$. Then $\delta/\ell \leq |y(t)| < r$ between T_2 and T_3, hence by (6.3.9) to (6.3.12) we have

$$\begin{aligned}
a(\varepsilon) &\leq a(|y(T_3)|) \leq V(T_3,y(T_3)) \\
&\leq V(T_2,y(T_2)) - \sigma(\delta/\ell)(T_3-T_2) + MK_1\, Q_{\delta/\ell}(T_2)[T_3-T_2+1] + \\
&\qquad + \sum_{n=N_1}^{\infty} K_1[\gamma_{\delta/\ell}(t_{n+1})h_{n+1}+M\, Q_{\delta/\ell}(t_{n+1})] \\
&\leq c(|y(T_2)|) + MK_1\, Q_{\delta/\ell}(T_1) + \sum_{n=N_1}^{\infty} K_1[\gamma_{\delta/\ell}(t_{n+1})h_{n+1}+MQ_{\delta/\ell}(t_{n+1})] \\
&\qquad + [MK_1\, Q_{\delta/\ell}(T_1) - \sigma(\delta/\ell)]\, [T_3-T_2] \\
&< c(\delta) + \frac{a(\varepsilon)}{3} + \frac{a(\varepsilon)}{3} \\
&< \frac{a(\varepsilon)}{3} + \frac{2a(\varepsilon)}{3} = a(\varepsilon),
\end{aligned}$$

a contradiction, proving (6.3.13). This proves that the trivial solution of (6.3.1) is eventually uniformly stable for the case $g(t,0) \equiv 0$. For the rest of the proof, choose $\delta_0 = \delta(r)$ and $T_0 = T_1(r)$. Fix $t_0 \geq T_0$ and $|x_0| < \delta_0$. Then (6.3.13) implies that $|y(t,t_0,x_0)| < r$ on $[t_0,\infty)$. Let $0 < \eta < r$ be given. Choose $\delta = \delta(\eta)$, $0 < \delta < \eta$, so that

$$3c(\delta) < a(\eta), \tag{6.3.14}$$

and $N_2 = N_2(\eta)$ so large that, $t_{N_2} \geq \tau_\delta+1$, that

$$\sum_{n=N_2}^{\infty} K_1[\gamma_\delta\,(t_{n+1})\,h_{n+1} + M\,Q_\delta(t_{n+1})] < \frac{a(\eta)}{3}\,. \qquad (6.3.15)$$

Then, choose $T_1 = T_1(\eta) \geq t_{N_2}$ such that

$$3MK_1\,Q_\delta(T_1) < \min\,(\sigma(\delta),\ a(\eta)). \qquad (6.3.16)$$

Select

$$T = \left[\frac{3c(r) + 3MK_1\,Q_\delta(1) + a(\eta) + 2T_1(\eta)\,\sigma(\delta)}{2\sigma(\delta)}\right] > T_1(\eta)$$

and it is clear that T depends only on η, not on t_o or x_o. We now claim that

$$|y(t,t_o,x_o)| < \delta \text{ for some } t_1 \in [t_o+T_1, t_o+T]. \qquad (6.3.17)$$

Suppose not. Then $\delta \leq |y(t,t_o,x_o)| < r$ on $[t_o+T_1, t_o+T]$. Let $y_o = y(t_o+T_1,t_o,x_o)$. Then by (6.3.9) to (6.3.16), we have

$$\begin{aligned} 0 < a(\delta) &\leq a(|y(t_o+T,\ t_o+T_1,y_o)|) \leq V(t_o+T,\ y(t_o+T)) \\ &\leq c(|y_o|) + [MK_1\,Q_\delta(t_o+T_1) - \sigma(\delta)][T-T_1] + MK_1\,Q_\delta(t_o+T_1) + \\ &\qquad + \sum_{n=N_2}^{\infty} K_1[\gamma_\delta(t_{n+1})\,h_{n+1} + MQ_\delta(t_{n+1})] \\ &< c(r) - \frac{2}{3}\,\sigma(\delta)\,[T-T_1] + MK_1\,Q_\delta(T_1) + \frac{a(\eta)}{3} = 0, \end{aligned}$$

a contradiction, proving (6.3.17). Thus by (6.3.13)

$$|y(t,t_1,y(t_1,t_o,x_o))| < \eta \text{ on } [t_1,\infty)$$

because $t_1 \geq t_o + T_1 > T_1$ and $|y(t_1,\ t_o,\ x_o)| < \delta$.

Hence, by uniqueness of solutions of (6.3.1) we have

$$|y(t,t_0,x_0)| < \eta \quad \text{for } t \geq t_0 + T.$$

Since η is arbitrary $|y(t,t_0,x_0)| \to 0$ as $t \to \infty$. Since T depends only on η and δ depends on ε , the trivial solution of (6.3.1) is eventually uniformly asymptotically stable if $g(t,0) \equiv 0$. This completes the proof.

Remark 6.3.1.

In view of remark 3.14.1 in [25] , in Theorem 6.3.1 the uniform asymptotic stability of the trivial solution of (6.3.2) can be replaced by eventual uniform asymptotic stability.

Remark 6.3.2.

If the perturbations in equation (6.3.1) are not impulsive so that the state of the system changes continuously with respect to time t, then the above result reduces to some of the results of [25] and [54] .

Example 6.3.1 .

Define $\gamma(t)$ on $[0,\infty)$ as follows. For each positive integer n, $\gamma(n) = 1$, otherwise $\gamma(t) = \dfrac{1}{1+t}$. Then $\gamma(t) \not\to 0$ as $t \to \infty$ and $\int_0^\infty \gamma(t)\, dt = \infty$. However,

$$G(t) = \int_t^{t+1} \gamma(s)\, ds \to 0 \text{ as } t \to \infty.$$

In this case $Q(t) = \log\left(1 + \frac{1}{t}\right)$ and $Q(t) \downarrow 0$ as $t \to \infty$.

Choose $u(t) = \sum_{k=1}^{\infty} u_k(t)$

where $$u_n(t) = 0 \text{ for } t < n^2,$$

$$= 1 \text{ for } t \geq n^2.$$

Then $h_n = 1$ and $t_n = n^2$ for $n = 1,2,3\ldots$ Clearly the discontinuities of u are isolated. Further

$$Q(t_n) = \log\left(1 + \frac{1}{n^2}\right) \text{ and } \gamma(t_n) = \frac{1}{1+n^2} .$$

Obviously the series $\sum_{n=1}^{\infty} \gamma(t_n)h_n$ and $\sum_{n=1}^{\infty} Q(t_n)$ are convergent.

Thus the conditions (A_1) to (A_4) are satisfied.

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